Eberhard Karls Universität Tübingen

Bachelorarbeit

About the Tropicalization of the Catalecticant and the Lüroth Invariant

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1 Introduction

Classical invariant theory played a central role in 19th-century algebra and geometry. For example the famous *Nullstellensatz* and *Basis Theorem* were first proven by Hilbert as lemmas in papers about invariant theory [PS08]. But what are invariants? Invariants are polynomials in the coefficients of homogeneous polynomials that satisfy the condition to be invariant under a certain group action of the special linear group on the set of polynomials. Especially invariants which give information about a polynomial were of great interest. Examples for such invariants are the Catalecticant and the Lüroth invariant.
More recently, at the end of the 20th-century, a new branch of mathematics came into play, tropical geometry. Tropical geometry is geometry based on the tropical semiring - which is the real numbers with infinity where the sum of two numbers is their minimum and the product is their sum. Since the beginning of the 21st-century, tropical geometry has undergone a rapid development and is still a topic of huge research interest today.
Of course, tropical geometry does not stand alone in mathematics. Among others, it has a direct connection to classical geometry over the field of Puiseux series. To every polynomial over the Puiseux series, there exists a tropical polynomial with a strong connection to it. This tropical polynomial arises from the normal one through a process called tropicalization.
In this Bachelor thesis we will investigate how two of the classical invariants - the already mentioned Catalecticant and Lüroth invariant - behave under tropicalization and if there are other criteria for the properties the classic invariants give information about.
After giving a more detailed introduction to the needed mathematics, we will consider the Catalecticant invariant. In 1852, James Joseph Sylvester proved that the Catalecticant of a polynomial vanishes if, and only if, the polynomial is a sum of unusual few powers of linear forms. However, we will show that this property does not hold for the tropicalized Catalecticant by constructing some examples.
In the next section, we will then see a necessary and sufficient criterion for the coefficients that plays the role of the Catalecticant in the tropical case. Unlike the Catalecticant, this criterion can be applied on polynomials of arbitrary degree and number of variables. Using this criterion, we will be able to tell if a polynomial with vanishing Catalecticant remains in its special form after tropicalizing it.
We will then move on to the second classical invariant we are going to consider, the Lüroth invariant. This invariant goes back to a paper from Jacob Lüroth published in 1868 [OS10]. The Lüroth of a quartic vanishes if, and only if, the quartic contains the vertices of a complete pentalateral i.e. the intersection points of five lines with exactly ten intersection points. In the case of classic geometry, the vanishing of the Lüroth invariant also gives information about the form of the polynomial. However, for tropical polynomials, this is not the case, as we will show.

2 Mathematical Background

2.1 Invariants

This section follows [Els15] and [PS08]. In both these texts all theorems and definitions are formulated over the complex number field $\mathbb{C}$, but in [KR84] the statements are shown more generally for any algebraically closed field $\mathbb{K}$ of characteristic 0.
For this section let $\mathbb{K}$ be an algebraically closed field of characteristic 0. The standard left action of $\mathrm{Gl}_n(\mathbb{K})$ on $\mathbb{K}^n$, $\mathrm{Gl}_n(\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n; (A, x) \mapsto Ax$, defines a right action on $\mathbb{K}[x_1, ..., x_n]$ by $f^A := f(Ax)$. If we denote the space of homogeneous polynomials of degree d as $\mathbb{K}^d[x_1, ..., x_n]$, we get the following definition.

Definition 2.1 (Invariant)**.** A mapping $I : \mathbb{K}^d[x_1, ..., x_n] \to \mathbb{K}$ which is a polynomial in the coefficients of $f \in \mathbb{K}^d[x_1, ..., x_n]$ is called an *invariant* if

$$I(f^A) = I(f) \det(A)^e$$

holds for some $e \in \mathbb{Z}$ and all $A \in \mathrm{Gl}_n(\mathbb{K}), f \in \mathbb{K}^d[x_1, ..., x_n]$.

Invariants are called invariants since they are invariant under the right action of $\mathrm{Sl}_n(\mathbb{K}) \subset \mathrm{Gl}_n(\mathbb{K})$.
An example for an invariant is the *Catalecticant invariant* for binary forms of even degree $2d$.

Definition 2.2 (Catalecticant Invariant C_{2d})**.** The *Catalecticant invariant* C_{2d} for a polynomial $f = \sum_{i=0}^{2d} c_i \binom{2d}{i} x^i y^{2d-i} \in \mathbb{K}^{2d}[x, y]$ is defined by

$$C_{2d}(c_0, ..., c_{2d}) := C_{2d}(f) := \det \begin{pmatrix} c_0 & c_1 & \cdots & c_d \\ c_1 & c_2 & \cdots & c_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_d & c_{d+1} & \cdots & c_{2d} \end{pmatrix}.$$

As already implicitly claimed by calling it so, the Catalecticant is an invariant.

Example 2.3. For the most simple case $d = 1$ we get

$$C_2(f) = C_2(c_0, c_1, c_2) = \det \begin{pmatrix} c_0 & c_1 \\ c_1 & c_2 \end{pmatrix} = c_0c_2 - c_1^2$$

where $f = c_0y^2 + 2c_1xy + c_2x^2 \in \mathbb{K}^2[x, y]$.

Example 2.4. If we set d as 2, the Catalecticant C_4 for a polynomial $f = c_4x^4 + 4c_3x^3 + 6c_2x^2y^2 + 4c_1xy^3 + c_0y^4 \in \mathbb{K}^4[x, y]$ is

$$C_4(c_0, ..., c_4) := C_4(f) := \det \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix} = c_0c_2c_4 - c_0c_3^2 - c_1^2c_4 + 2c_1c_2c_3 - c_2^3.$$

Additionally to being an invariant, the zero set of the Catalecticant can also be characterized by requiring the following condition for the polynomials.

Theorem 2.5 (Sylvester 1852)**.** *A binary form of degree* $2d$ *can be split into a sum of* d *or less powers of linear forms if, and only if, its Catalecticant is zero. Or, written in symbols,*

$$\{f \in \mathbb{K}^{2d}[x,y] \mid C_{2d}(f) = 0\}$$
$$= \{f \in \mathbb{K}^{2d}[x,y] \mid \exists \mu_i, \nu_i \in \mathbb{K}, i = 1, ..., d : f = \sum_{j=1}^{d} (\mu_i x + \nu_i y)^{2d}\}. \quad (1)$$

A (modern version of the) proof of this theorem can be found in [Sel].

Example 2.6. For $d = 1$ we know from example 2.3 that

$$\{c_0 y^2 + 2c_1 xy + c_2 x^2 \in \mathbb{K}^2[x,y] \mid c_0 c_2 = c_1^2\} = \{(\mu x + \nu y)^2 \in \mathbb{K}^2[x,y] \mid \mu, \nu \in \mathbb{K}\}.$$

By expanding $(\mu x + \nu y)^2 = \nu^2 y^2 + 2\nu\mu xy + \mu^2 x^2$, we see that the set on the right is clearly contained in the left one.
Let $c_0 c_2 = c_1^2$ and $c_0, c_2 \neq 0$. We can then choose μ as a square root of c_2, define $\nu = c_1 \cdot \mu^{-1}$ and get $c_0 y^2 + 2c_1 xy + c_2 x^2 = \frac{c_1^2}{c_2} y^2 + 2c_1 \cdot \mu^{-1} \cdot \mu + c_2 x^2 = \nu^2 y^2 + 2\nu\mu xy + \mu^2 x^2 = (\mu x + \nu y)^2$. If c_2 or c_0 is zero, c_1 is zero, too, and the existence of μ and ν is clear. So the other inclusion also holds.
Let $\mathbb{K} = \mathbb{C}$ and $f = y^2 + 2ixy - x^2$. Then we get two possible pairs (μ, ν):

(i) $\mu = i, \nu = 1$: $(ix + y)^2 = y^2 + 2ix + i^2 x^2 = f$ and

(ii) $\mu = -i, \nu = -1$: $(-ix - y)^2 = (-1)^2 y^2 + 2(-1)(-i)xy + (-i)^2 x^2 = f$,

which demonstrates that μ and ν are not uniquely determined.
In fact, since we are only considering polynomials of even degree, we can always substitute μ_i, ν_i by $-\mu_i, -\nu_i$ for an arbitrary $i \in \{1, ..., d\}$ and still get a representation of the same polynomial.

Example 2.7. Let now $d = 2$ and again $\mathbb{K} = \mathbb{C}$. Consider the polynomial $f = 2x^4 + 4x^3 y + 30x^2 y^2 + 28xy^3 + 17y^4 \in \mathbb{C}^4[x,y]$. To see the coefficients needed to compute the Catalecticant, we rewrite f as

$$f = 2x^4 + 4 \cdot 1 \cdot x^3 y + 6 \cdot 5x^2 y^2 + 4 \cdot 7xy^3 + 17y^4.$$

We get

$$C_4(f) = C_4(17,7,5,1,2) = \det \begin{pmatrix} 17 & 7 & 5 \\ 7 & 5 & 1 \\ 5 & 1 & 2 \end{pmatrix}$$
$$= \det \begin{pmatrix} 10 & 2 & 4 \\ 7 & 5 & 1 \\ 5 & 1 & 2 \end{pmatrix} = 2 \det \begin{pmatrix} 5 & 1 & 2 \\ 7 & 5 & 1 \\ 5 & 1 & 2 \end{pmatrix} = 0.$$

By theorem 2.5 follows that there exist $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{C}$ such that $f = (\mu_1 x + \nu_1 y)^4 + (\mu_2 x + \nu_2 y)^4$.
Explicitly, it holds $f = (x - y)^4 + (x + 2y)^4$.

If we identify a polynomial $f = \sum_{i=0}^{2d} c_i \binom{2d}{i} x^i y^{2d-i} \in \mathbb{K}^{2d}[x,y]$ with the point $(c_0, ..., c_d) \in \mathbb{K}^{2d+1}$, we get to write the left part of equation (1) as $V(C_{2d})$. Instead of $(c_0, ..., c_{2d}) \in V(C_{2d})$ we will often write $f \in V(C_{2d})$.
Our main aim in this Bachelor thesis is to prove a similar statement for tropical polynomials. Before approaching this, we first have to clarify what those are.

2.2 Tropical Mathematics and Polynomials

This subsection follows [SS09].
To define tropical polynomials, we need to introduce the tropical semiring.

Definition 2.8 (Tropical Semiring)**.** We define the *tropical semiring* $(\mathbb{T}, \oplus, \odot)$ as the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ of the real numbers $\mathbb{R}$ with an extra symbol ∞ and the arithmetic operations $x \oplus y := \min(x, y)$, $x \odot y := x + y$.
We set $x \oplus \infty = \infty \oplus x = x$ and $\infty \odot x = x \odot \infty = \infty$ for all $x \in \mathbb{T}$.

Obviously, the symbol ∞ behaves like and intuitively represents positive infinity. In other texts, the tropical semiring is often isomorphically defined as $(\mathbb{R} \cup \{-\infty\}, \max, +)$. But for our purposes, the definition given above is more practical. As usual we will refer to the tropical semiring by only writing $\mathbb{T}$.
In our chosen definition, the neutral element regarding $\oplus$ is ∞, regarding $\odot$ it is 0. The associative, distributive and commutative laws for $\oplus$ and $\odot$ hold in the tropical semiring. Since there are no inverse elements with respect to tropical addition for any element other than ∞, $(\mathbb{T}, \oplus, \odot)$ is not a ring. Throughout this text, we will only use $\oplus$ and $\odot$ to avoid getting into trouble due to not-existing inverses. If a fraction or a minus occurs, we mean the quotient or respectively the additive inverse in the real numbers. However, $(\mathbb{T} \backslash \{\infty\}, \odot) = (\mathbb{R}, +)$ is a group. The tropical multiplicative inverse of an element $x \in \mathbb{T} \backslash \{\infty\}$ is $-x$, since $x \odot (-x) = x + (-x) = 0$.
The tropical n-th power, $n \in \mathbb{N}$ a natural number, of an element $x \in \mathbb{T}$ is defined as $\underbrace{x \odot ... \odot x}_{n \text{ times}}$ and will be written as x^n. For any $x \in \mathbb{T} \backslash \{\infty\}$ and $n \in \mathbb{N}$ holds tropically $(\frac{x}{n})^n = x$, so we always have this kind of an "n-th root". Wherever it is possible without causing confusion, we will write $x^i y^j$ instead of $x^i \odot y^j$ to shorten the length of expressions.
An other unfamiliar property of the tropical semiring is that the tropical addition is idempotent; for any $x \in \mathbb{T}$ holds $x \oplus x \oplus ... \oplus x = \min\{x, x, ..., x\} = x$. We will use the symbol $\bigoplus$ for tropical sums instead of $\sum$. Moreover, the *Freshman's Dream* holds [MS15].

Lemma 2.9 (Freshman's Dream)**.** *Let $x, y \in \mathbb{T}$ be tropical numbers and $n \in \mathbb{N}$ be a natural number. Then*

$$(x \oplus y)^n = \bigoplus_{i=0}^{n} x^i \odot y^{n-i} = x^n \oplus y^n$$

holds.

Proof. We compute $(x \oplus y)^n = n \cdot \min(x, y) = \min(nx, ny) = x^n \oplus y^n$. The third expression in the equation above is attained by expanding $(x \oplus y)^n$ as usual while paying attention to the idempotency of the tropical semiring with respect to tropical addition. □

This equality will play an important role in the following sections.
Now that we know what the tropical semiring is, we can define tropical polynomials. To do so, let $x_1, ..., x_n$ be variables which represent elements in $\mathbb{T}$. A *tropical monomial* is any product of these variables, for example $x_1^3 x_3^2 x_5$. Since 0 is the neutral element with respect to tropical multiplication, we write x instead of $0 \odot x$. It is important to keep that in mind.

Further we write x^i instead of $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$. By e_k we mean the k-th standard base vector. For example $x^{(3,0,2,0,1)}$ will stand for $x_1^3x_3^2x_5$ and $x^{5e_3} = x_3^5$.

Definition 2.10 (Tropical Polynomial)**.** A *tropical polynomial* F is a finite linear combination of tropical monomials

$$\begin{aligned} F(x_1,...,x_n) &= a\odot x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \oplus b\odot x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n} \oplus ... \\ &= \min(a + i_1x_1 + ... + i_nx_n, b + j_1x_1 + ... + j_nx_n, ...). \end{aligned}$$

We will write $\mathbb{T}[x_1,...,x_n]$ for the tropical polynomials in $x_1,...,x_n$ and $\mathbb{T}^d[x_1,...,x_n]$ for the homogeneous tropical polynomials of degree d.

Degree, addition and multiplication of tropical polynomials are defined analogously to common polynomials.

Distinct tropical polynomials can represent the same function. For example $x^2 \oplus 17\odot x \oplus 2 = x^2 \oplus 1\odot x \oplus 2 = (x \oplus 1)^2$ holds for all $x \in \mathbb{T}$. *In this text we will always mean the polynomial function.* We will write $F = G$ for $F, G \in \mathbb{T}[x_1,...,x_n]$ if $F(x)$ and $G(x)$ coincide for all $x \in \mathbb{T}^n$.

2.3 Tropicalization

Definitions and theorems are taken from [RGST05], [SS09] and [MS15].
For a polynomial $f \in \mathbb{T}[x_1,...,x_n]$ its algebraic variety is simply defined as the zero set of f in the affine space $\mathbb{A}^n_{\mathbb{K}} = \mathbb{K}^n$, $V(f) := \{p \in \mathbb{A}^n_{\mathbb{K}} \mid f(p) = 0\}$. For tropical polynomials it is a bit more complicated. For our case it will be sufficient to only define tropical hyperplanes, tropical varieties defined by only one polynomial.

Definition 2.11 (Tropical Hyperplane)**.** The *tropical hyperplane* $V^{\text{trop}}(F)$ of a tropical polynomial $F \in \mathbb{T}[x_1,...,x_n]$ is defined as the set of points x in $\mathbb{R}^n$ where the minimum is attained at least twice.

Even though a point $P \in V^{\text{trop}}(F)$ does not annihilate F, we will still call $V^{\text{trop}}(F)$ the zero set of F. Tropical hyperplanes behave similarly to common ones: Let F, G be elements of $\mathbb{T}[x_1,...,x_n]$. If $P \in V^{\text{trop}}(F)$ then $P \in V^{\text{trop}}(F\odot G)$. Also, if $P \in V^{\text{trop}}(F)$ and $P \in V^{\text{trop}}(G)$ then $P \in V^{\text{trop}}(F \oplus G)$. To get an idea how tropical hyperplanes look like, we will give some examples. All plots of tropical varieties were done using Matlab [MAT20]. The code can be found in the appendix, code 6.6.

Example 2.12. Consider $F = x \oplus y \oplus 0 = \min(x, y, 0)$. By definition 2.11 we get $V^{\text{trop}}(F) = \{(x, y) \in \mathbb{R}^2 \mid x = y \leq 0 \text{ or } x = 0 \leq y \text{ or } y = 0 \leq x\}$. This set is illustrated in figure 1.
As we are used to from "normal" zero sets, the zero set of a power of a tropical polynomial F is the same as the zero set of F, $V^{\text{trop}}(F^d) = V^{\text{trop}}(F)$ for any natural number $d > 0$. In conclusion the left graphic in figure 1 also represents $V^{\text{trop}}(x^2 \oplus y^2 \oplus 0)$.
Since $G = x^4 \oplus x \oplus y^2 \oplus y^4 \oplus 0 = \min(4x, x, 2y, 4y, 0) = \min(4x, 4y, 0)$ and F represent the same function, their zero sets coincide, of course.

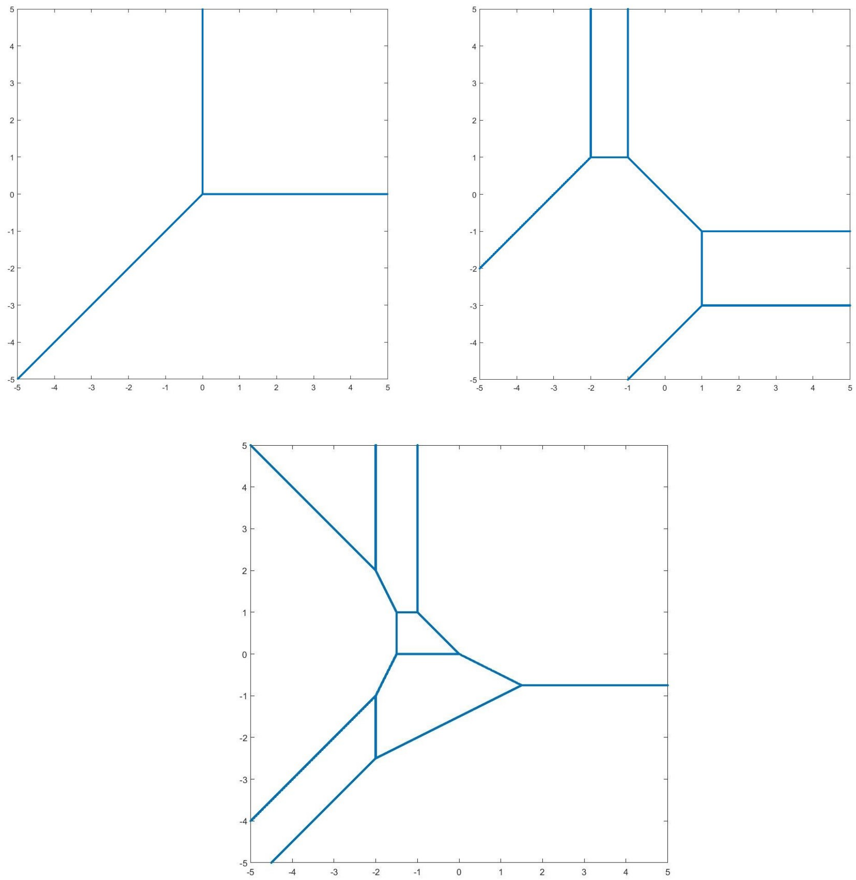

Figure 1: On the left: $V^{\text{trop}}(x \oplus y \oplus 0) = V^{\text{trop}}(x^2 \oplus y^2 \oplus 0)$
$= V^{\text{trop}}(x^4 \oplus x \oplus y^2 \oplus y^4 \oplus 0)$.
On the right: $V^{\text{trop}}(2{\odot}x^2 \oplus x \oplus 3{\odot}y^2 \oplus y \oplus (-1){\odot}xy \oplus -1)$
At the bottom:
$V^{\text{trop}}(2{\odot}x^2 \oplus x \oplus 2{\odot}y^4 \oplus 2{\odot}x^3y \oplus -1{\odot}xy \oplus -1{\odot}xy^2 \oplus 1{\odot}x^2y^2 \oplus -1)$

Again, if for a tropical polynomial $F = \bigoplus_{i=0}^{2d} c_i{\odot}x^i y^{2d-i}$ holds $(c_0, ..., c_{2d}) \in V^{\text{trop}}(G)$ for some $G \in \mathbb{T}[x_0, ..., x_{2d}]$ we will write $F \in V^{\text{trop}}(G)$.
Definition 2.11 will make more sense as soon as we introduce the concept of tropicalization. But before that, we need to define a special field.

Definition 2.13 (Puiseux Series)**.** The field of *Puiseux series* with coefficients in the complex numbers $\mathbb{C}$ are formal power series

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + ...$$

where c_i are complex numbers and $a_1 < a_2 < a_3 < ...$ are rational numbers with a common denominator. We write $\mathfrak{C}$ for the field of Puiseux series over $\mathbb{C}$.
The set of such formal power series but with coefficients only in $\mathbb{R}$ is a subfield of $\mathfrak{C}$. We write $\mathfrak{R}$ for this field.

Theorem 2.14. *The field $\mathfrak{C}$ of Puiseux series with coefficients in the complex numbers is algebraically closed.*

Moreover, $\mathfrak{C}$ and $\mathfrak{R}$ are fields with valuation.

Definition 2.15 (Valuation)**.** Let $\mathbb{K}$ be a field. A *valuation* on $\mathbb{K}$ is a function $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ satisfying the following three conditions:

(1) $\mathrm{val}(a) = \infty$ if, and only if, $a = 0$

(2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ and

(3) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in \mathbb{K} \backslash \{0\}$.

The valuation on $\mathfrak{C}$ is defined via $c = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + ... \mapsto a_1$ for $c \neq 0$ and $0 \mapsto \infty$.
A valuation map on a field $\mathbb{K}$ extends naturally to $\mathbb{K}^n$ by applying it to every entry of a vector in $\mathbb{K}^n$.
The following lemma holds for all fields with valuation [MS15].

Lemma 2.16. *Let $\mathbb{K}$ be a field with valuation and $a, b \in \mathbb{K}$.*
If $\mathrm{val}\, a \neq \mathrm{val}\, b$, then $\mathrm{val}(a + b) = \min(\mathrm{val}\, a, \mathrm{val}\, b)$.

Now that we know what the Puiseux series and their valuation are, we can finally make the connection between "normal" algebraic hyperplanes and tropical ones.

Definition 2.17 (Tropicalization)**.** The *tropicalization* of a polynomial $f = \sum_i c_i x^i \in \mathfrak{C}[x_1, ..., x_n]$ is defined as $\mathrm{trop}\, f := \bigoplus_i \mathrm{val}(c_i) \odot x^i$.

The map $\mathrm{trop} : \mathfrak{C}[x_1, ..., x_n] \to \mathbb{T}[x_1, ..., x_n]; f \mapsto \mathrm{trop}\, f$ is obviously far away from being injective. It also is not a homomorphism, since for example $\mathrm{trop}((-x) + (t + 1)x) = \mathrm{trop}\, tx = 1 \odot x \neq 0 \odot x = 0 \odot x \oplus 1 \odot x = \mathrm{trop}(-x) \oplus \mathrm{trop}((t + 1)x)$ for $x \neq \infty$.
However, it holds $\mathrm{trop}(c_i x^i + c_j x^j) = \mathrm{trop}(c_i x^i) \oplus \mathrm{trop}(c_j x^j)$ if $i \neq j$.

Example 2.18. The tropicalization of the two polynomials $f = tx$ and $g = (t^2 + t)x$ in $\mathfrak{C}[x]$ coincides:

$$\mathrm{trop}\, tx = \mathrm{val}(t) \odot x = 1 \odot x = \mathrm{val}(t^2 + t) \odot x = \mathrm{trop}(t^2 + t)x.$$

In particular, this example shows that trop is not injective.

Example 2.19. Consider the Catalecticant for $d = 2$:

$$C_4(c_0, ..., c_4) = c_0 c_2 c_4 - c_0 c_3^2 - c_1^2 c_4 + 2 c_1 c_2 c_3 - c_2^3.$$

Its tropicalization is

$$\mathrm{trop}\, C_4(c_0, ..., c_4) = 0 \odot c_0 c_2 c_4 \oplus 0 \odot c_0 c_3^2 \oplus 0 \odot c_1^2 c_4 \oplus 0 \odot c_1 c_2 c_3 \oplus 0 \odot c_2^3,$$

since $\mathrm{val}(1) = \mathrm{val}(2) = 0$.

Remark 2.20. Since $\mathrm{val}(c) = 0$ for all $c \in \mathbb{C}\backslash\{0\} \subset \mathfrak{C}$ and $\mathrm{val}(a \cdot b) = \mathrm{val}(a) + \mathrm{val}(b)$ for all $a, b \in \mathfrak{C}$ holds, we get

$$\mathrm{trop}(\sum_{i=0}^{2d} c_i \binom{2d}{i} x^i y^{2d-i}) = \bigoplus_{i=0}^{2d} \mathrm{val}(c_i) \odot x^i y^{2d-i},$$

so the binomial coefficients which are added in the representation of polynomials in the context of invariants are not relevant for tropicalization.

The following theorem gives us the connection mentioned above.

Theorem 2.21 (Kapranov)**.** *For $f \in \mathfrak{C}[x_1, ..., x_n]$ the tropical hyperplane $V^{\mathrm{trop}}(\mathrm{trop}\, f)$ in $\mathbb{R}^n$ and the topological closure of $\{(\mathrm{val}\, y_1, ..., \mathrm{val}\, y_n) \mid y \in V(f)\}$ in $\mathbb{R}^n$ coincide.*

Taking the topological closure is necessary since the image of $(\mathfrak{C}\backslash\{0\})^n$ under the valuation map val lies only in $\mathbb{Q}^n$.
Since this theorem does not include the case that entries of a point in the variety are zero, we need the following corollary.

Corollary 2.22. *Let f be an element in $\mathfrak{C}[x_1, ..., x_n]$ and $P = (p_1, ..., p_n)$ be an arbitrary point in $V(f)$ with at least one entry unequal to zero. Then $\mathrm{val}\, P$ lies in $V^{\mathrm{trop}}(\mathrm{trop}\, f)$.*

Proof. Let f and P be as stated above. Without loss of generality, let the zero entries of $P \in V(f)$ be $p_1, ..., p_r$. Define $d = \deg f$ to be the degree of f. Let f_0 be the polynomial defined as the sum of all monomials in f in which $x_1, ..., x_r$ do not occur and $f_1 := f - f_0$.
If $P = 0$, it follows that $f_0 = 0$. In conclusion, $\mathrm{trop}\, f(\infty, ..., \infty) = \infty$. Hence, all entries in the minimum are equal, they are ∞.
Let now $P \neq 0$. If f_0 is unequal to zero, we know that $(p_{r+1}, ..., p_n) \in V(f_0)$ and $\mathrm{trop}\, f_0 \neq \infty$. Since $p_{r+1}, .., p_n \neq 0$ holds, we get by Kapranov's theorem 2.21 that $\mathrm{val}(p_{r+1}, ..., p_n) \in V^{\mathrm{trop}}(\mathrm{trop}\, f_0)$. Further we know that $\mathrm{trop}\, f = \mathrm{trop}(f_0) \oplus \mathrm{trop}(f_1)$. It follows that if we consider $\mathrm{trop}\, f(\infty, ..., \infty, \mathrm{val}\, p_{r+1}, ..., \mathrm{val}\, p_n)$ the minimum is attained twice:
Evaluating $\mathrm{trop}(f_1)$ for $x = (\infty, ..., \infty, \mathrm{val}\, p_{r+1}, ..., \mathrm{val}\, p_n)$ gives ∞ in all entries of the minimum and $\mathrm{val}(p_{r+1}, .., p_n) \in V^{\mathrm{trop}}(\mathrm{trop}\, f_0)$ with $\mathrm{trop}\, f_0(\mathrm{val}\, P) < \infty$.
If f_0 is equal to zero, it holds $\mathrm{trop}\, f(\mathrm{val}\, P) = \infty$. Hence, all entries in the minimum are equal, they are ∞. □

3 Tropicalization of the Catalecticant

Kapranov's theorem 2.21 can of course also be applied to the zero set of the Catalecticant. In this section we will give counterexamples to the assumption that Sylvester's theorem 2.5 holds for the tropicalized Catalecticant in tropical mathematics. This statement gets more clearly formulated in the next proposition. To do so, we need the following lemma.

Lemma 3.1. *A homogeneous tropical polynomial $F \in \mathbb{T}^d[x_1, ..., x_n]$ is a sum of m d-th powers of linear forms if, and only if, it is a d-th power of a linear form.*

Proof. Let $F = \bigoplus_{i=1}^{m}(c_{1,i}\odot x_1 \oplus ... \oplus c_{n,i}\odot x_n)^d$. We compute

$$\begin{aligned}
F =& \bigoplus_{i=1}^{m}(c_{1,i}\odot x_1 \oplus ... \oplus c_{n,i}\odot x_n)^d \\
=& \bigoplus_{i=1}^{m}(d \cdot c_{1,i}\odot x_1^d \oplus ... \oplus d \cdot c_{n,i}\odot x_n^d) \\
=& (\bigoplus_{i=1}^{m} d \cdot c_{1,i})\odot x_1^d \oplus ... \oplus (\bigoplus_{i=1}^{m} d \cdot c_{n,i})\odot x_n^d \\
=& \left((\bigoplus_{i=1}^{m} c_{1,i})\odot x_1 \oplus ... \oplus (\bigoplus_{i=1}^{m} c_{n,i})\odot x_n \right)^d .
\end{aligned}$$

The converse is obviously true; set $m = 1$. □

In particular, this means that the property which the Catalecticant checks a polynomial for is a lot more special in the tropical than in the usual case. In combination with the Freshman's Dream, lemma 2.9, we see that it is sufficient to show that the minimum of a tropical polynomial is only attained by terms $c_i\odot x_i^d$ to prove that a tropical polynomial is a d-th power of a linear form. The following notation will be useful.

Definition 3.2 (A_{2d} and B_{2d})**.** We set

$$\begin{aligned}
A_{2d} :=& V(C_{2d}) \\
=& \{f \in \mathfrak{C}^{2d}[x, y] \mid \exists \mu_i, \nu_i, \in \mathfrak{C}, i = 1, ..., d : f = \sum_{i=1}^{d}(\mu_i x + \nu_i y)^{2d}\}
\end{aligned}$$

to be the set of bivariate polynomials over $\mathfrak{C}$ which can be written as the sum of d $2d$-th powers of linear forms.
The tropical version of this set will be denoted

$$B_{2d} := \{F \in \mathbb{T}^{2d}[x, y] \mid \exists \mu, \nu \in \mathbb{T} : F = (\mu\odot x \oplus \nu\odot y)^{2d}\}.$$

In words, B_{2d} is the set of $2d$-th powers of tropical linear forms.

We can now state our claim that the tropicalized Catalecticant has no such practical property as the normal one.

Proposition 3.3. *The property of the Catalecticant C_{2d} stated in theorem 2.5 does not hold for its tropicalization if $d > 1$.*
Precisely, for any $d > 1$ there is a polynomial $f \in A_{2d}$, such that $\operatorname{trop} f \notin B_{2d}$, i.e. $\overline{\operatorname{trop} A_{2d}} = V^{\text{trop}}(\operatorname{trop} C_{2d}) \not\subseteq B_{2d}$, and there is a polynomial $f \notin A_{2d}$ for which $\operatorname{trop} f \in B_{2d}$ holds.

Proof. First, we will give two examples of polynomials $f, g \in \mathfrak{C}^{2d}[x, y]$ which lie in A_{2d} but for which their tropicalization does not lie in B_{2d}.
Let d be greater than 1 and

$$f = \sum_{i=1}^{2d} \binom{2d}{i} x^i y^{2d-i}.$$

We compute

$$C_{2d}(f) = C_{2d}(0,1,...,1) = \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = 0,$$

since the matrix has at least three columns. If we apply the tropicalization map to f, we get the tropical polynomial

$$\operatorname{trop} f = \infty \odot y^{2d} \oplus \bigoplus_{i=1}^{2d} 0 \odot x^i y^{2d-i}.$$

For $x < y$ the minimum in $\operatorname{trop} f$ is then attained by $0 \odot x \odot y^{2d-1}$. It follows $\operatorname{trop} f \notin B_{2d}$.
However, it is not necessary that one of the coefficients becomes ∞ under tropicalization: We define ζ to be a $2d$-th root of $t^q - t^{2d}$, $2d < q \in \mathbb{Q}$. Let again $d > 1$. Such a ζ exists since $\mathfrak{C}$ is algebraically closed (theorem 2.14). If we consider the polynomial

$$g = (x + \zeta y)^{2d} + (tx + ty)^{2d} \in A_{2d},$$

using the binomial theorem we compute

$$g = t^q y^{2d} + \sum_{i=1}^{2d} \binom{2d}{i} (\zeta^{2d-i} + t^{2d}) x^i y^{2d-i}.$$

To be able to compute the tropicalization, we need to find the valuation of the coefficients of g. We know that $2d = \operatorname{val}(t^q + t^{2d}) = \operatorname{val}(\zeta^{2d}) = 2d \cdot \operatorname{val}(\zeta)$, in conclusion $\operatorname{val}(\zeta) = 1$. It holds $\operatorname{val}(t^{2d}) = 2d \operatorname{val}(t) = 2d$ and $\operatorname{val}(\zeta^{2d-i}) = (2d-i)\operatorname{val}(\zeta) = 2d - i$. By lemma 2.16 we get, for $i \neq 0$,

$$\operatorname{val}(\zeta^{2d-i} + t^{2d}) = \min\{\operatorname{val}(\zeta^{2d-i}), \operatorname{val}(t^{2d})\} = \min\{2d - i, 2d\} = 2d - i$$

and obtain

$$\operatorname{trop} g = q \odot y^{2d} \oplus \bigoplus_{i=1}^{2d} (2d - i) \odot x^i y^{2d-i}.$$

Evaluating $\operatorname{trop} g$ for $x = 0$ and $y = -2$, we get $\operatorname{trop} g(0, -2)$

$$\begin{aligned} &= \min(q + 2d \cdot (-2), (2d-1) + (2d-1) \cdot (-2), (2d-2) + (2d-2) \cdot (-2), ..., 0) \\ &= \min(q - 4d, 1 - 2d, 2 - 2d, ..., -1, 0). \end{aligned}$$

We see that the minimum is not attained by $c_0 \odot y^{2d}$ or $c_{2d} \odot x^{2d}$ if we choose q to be greater than $4d$. It follows that $\operatorname{trop} g \notin B_{2d}$ and hence $\overline{\operatorname{trop} A_{2d}} = V^{\text{trop}}(\operatorname{trop} C_{2d}) \nsubseteq B_{2d}$.

Now we will give an example for a polynomial $h \in \mathfrak{C}^{2d}[x, y]$ satisfying $h \notin A_{2d}$ and $\operatorname{trop} h \in B_{2d}$.
Consider the polynomial

$$k = (x + y)^{2d} + (x + y)^{2d} = \sum_{i=0}^{2d} 2 \binom{2d}{i} x^i y^{2d-i} \in A_{2d}.$$

Its tropicalization is

$$\operatorname{trop} k = \bigoplus_{i=0}^{2d} x^i y^{2d-i} = (x \oplus y)^{2d},$$

in particular it is an element of B_{2d}. The second equality holds by lemma 2.9. If we now define $h = k + tx^d y^d$, it holds $\operatorname{trop} h = \operatorname{trop} k \in B_{2d}$ since $\operatorname{val}(2+t) = 0 = \operatorname{val}(2)$. But taking a closer look at

$$C_{2d}(h) = \det \begin{pmatrix} 2 & 2 & \cdots & 2 & 2+t \\ 2 & \iddots & \iddots & \iddots & 2 \\ \vdots & \iddots & 2+t & \iddots & \vdots \\ 2 & \iddots & \iddots & \iddots & 2 \\ 2+t & \cdots & \cdots & 2 & 2 \end{pmatrix},$$

we notice that this determinant is a polynomial in t of degree d. Especially, $C_{2d}(h)$ is not equal to zero, so h is not an element of A_{2d}. ☐

Remark 3.4. The examples in this proof also demonstrate that restricting to polynomials with only coefficients in $\mathfrak{R}$ is not sufficient to keep the special property of the Catalecticant preserved through tropicalization.

However, one inclusion from Sylvester's theorem stays true in tropical mathematics. Remember that we identified all tropical polynomials which define the same function.

Proposition 3.5. *It holds* $V^{\text{trop}}(\operatorname{trop} C_{2d}) = \overline{\operatorname{trop} A_{2d}} \supset B_{2d}$ *for* $d > 1$.

Proof. Let $F = (\mu \odot x \oplus \nu \odot y)^{2d} \in B_{2d}$, without loss of generality both μ and ν not ∞. Then clearly $F = 2d\mu \odot x^{2d} \oplus 2d\nu \odot y^{2d}$ arises from $f := t^{2d\mu} x^{2d} + t^{2d\nu} y^{2d}$ through tropicalization. Since $d > 1$,

$$C_{2d}(f) = \det \begin{pmatrix} t^{2d\mu} & 0 & \cdots & 0 \\ 0 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & 0 \\ 0 & \cdots & 0 & t^{2d\nu} \end{pmatrix}$$

vanishes due to a zero-line in the matrix and hence $f \in A_{2d}$. Finally, Kapranov's theorem gives us that $\operatorname{trop} f = F \in \overline{\operatorname{trop} A_{2d}}$. ☐

That the inclusion does not hold in the other direction was proven in proposition 3.3. Anyway, we will give an other, more direct, example for that.

Example 3.6. Consider the tropical polynomial $F = 0 \odot x^4 \oplus (-1) \odot x^3 y \oplus (-1) \odot xy^3 \oplus 0 \odot y^4$. Clearly, no term of this polynomial is redundant for the defined function and hence it holds $F \notin B_4$. However, we get

$$\begin{aligned} &\operatorname{trop} C_4(0, -1, \infty, -1, 0) \\ &\quad = \min(0 + \infty + 0, 0 + 2 \cdot (-1), 2 \cdot (-1) + 0, (-1) + \infty + (-1), 3 \cdot \infty) \\ &\quad = \min(\infty, -2, -2, \infty, \infty). \end{aligned}$$

In conclusion, it holds $F \in V^{\text{trop}}(\operatorname{trop} C_4) \backslash B_4 = \overline{\operatorname{trop} A_4} \backslash B_4$.

Remark 3.7. The proposition stated above is only true if we identify tropical polynomials which represent the same function. Whether the tropicalized Catalecticant has F as a root or not *does* depend on its chosen representation as a polynomial. We will give an example later in the text, see example 4.5.

In corollary 4.7 we will see that $V^{\text{trop}}(\operatorname{trop} C_2) = \overline{\operatorname{trop} A_2} = B_2$.

4 Approach to a tropical Catalecticant

As shown in the previous section, the Catalecticant does not translate to tropical mathematics by simply tropicalizing it. However, our goal for this section is to prove a different criterion for a tropical polynomial to be a power of a linear form.
We will then, in the second subsection, apply this criterion on tropical polynomials which arise from polynomials out of the zero set of the Catalecticant.

4.1 A general Criterion

The following proposition is useful to simplify tropical polynomials. Additionally, it gives us the desired criterion as a direct conclusion.

Proposition 4.1. *Let $F \in \mathbb{T}^d[x_1, ..., x_n]$ and $c_i = c_{(i_1,...,i_n)}$ be the coefficient of $x^i = x_1^{i_1} \cdot ... \cdot x_n^{i_n}$.*
Then the term $c_i \odot x^i$, $i \neq de_1, ..., de_n$, is not relevant for the function defined by F if $c_{(i_1,...,i_n)} \geq i_1 \frac{c_{de_1}}{d} + ... + i_n \frac{c_{de_n}}{d}$.

Proof. Let x be a vector in $\mathbb{T}^n$ and $j \in \{1, ..., n\}$ such that $x_j + \frac{c_{de_j}}{d} = \min_{k \in \{1,...,n\}}(x_k + \frac{c_{de_k}}{d})$. We compute

$$\begin{aligned}
& c_i + i_1 x_1 + ... + i_n x_n \\
\geq\ & i_1 \frac{c_{de_1}}{d} + ... + i_n \frac{c_{de_n}}{d} + i_1 x_1 + ... + i_n x_n \\
=\ & \sum_{k=1}^{n} i_k (x_k + \frac{c_{de_k}}{d}) \\
\geq\ & \sum_{k=1}^{n} i_k (x_j + \frac{c_{de_j}}{d}) = (\sum_{k=1}^{n} i_k) \cdot (x_j + \frac{c_{de_j}}{d}) \\
=\ & d(x_j + \frac{c_{de_j}}{d}) \\
=\ & c_{de_j} + d x_j.
\end{aligned}$$

We see that $c_i + i_1 x_1 + ... + i_n x_n$ is never the single smallest entry in the minimum and hence not relevant for the function defined by F. □

Theorem 4.2. *Let $F \in \mathbb{T}^d[x_1, ..., x_n]$ and $c_i = c_{(i_1,...,i_n)}$ be the coefficient of $x^i = x_1^{i_1} \cdot ... \cdot x_n^{i_n}$.*
Then F is the d-th power of a tropical linear form if, and only if, it holds

$$c_{(i_1,...,i_n)} \geq \frac{c_{de_1}}{d} i_1 + ... + \frac{c_{de_n}}{d} i_n \tag{2}$$

for all indices i.

Proof. If the inequalities hold, applying proposition 4.1 gives us that $F = \bigoplus_{i=1}^{n} c_{de_i} \odot x_i^d$, so $F = (\bigoplus_{i=1}^{n} \frac{c_{de_i}}{d} \odot x_i)^d$.

Let now $F = (\bigoplus_{i=1}^{n} c_{de_i} \odot x_i)^d = \bigoplus_{i=1}^{n} (d \cdot c_{de_i} \odot x_i^d)$. Then $c_i = \infty$ for $i \neq de_k$, in particular all inequalities (2) hold. $\square$

Example 4.3. Consider $F = 2\odot y^4 \oplus 2\odot x^3 y \oplus 1\odot x^2 y^2 \oplus 2\odot xy^3 \oplus 5\odot z^4$. If we write F as $y^2 \odot \underbrace{(2\odot y^2 \oplus 2\odot xy \oplus 1\odot x^2)}_{=:G} \oplus 2\odot x^3 y \oplus 5\odot z^4$, and apply proposition 4.1 to the tropical polynomial G, we see that the term $2\odot xy^3$ in F is actually redundant: It holds $2 \geq \frac{2}{2} + \frac{1}{2}$.
This example demonstrates that the criterion in proposition 4.1 is not necessary, since $c_{(1,3,0)} = 2 \not\geq \infty = 1\frac{c_{4e_1}}{4} + 3\frac{c_{4e_2}}{4} + 0\frac{c_{4e_3}}{4}$.

Example 4.4. Let $F = 4\odot x^4 \oplus 0\odot y^4 \oplus (-4)\odot z^4 \oplus 3\odot x^3 y \oplus 1\odot x^2 z^2 \oplus (-2)\odot yz^3$. The inequalities of theorem 4.2 are $c_{(i,j,k)} \geq i\frac{4}{4} + j\frac{0}{4} + k\frac{-4}{4} = i - k$. It holds

$$\begin{aligned} c_{(3,1,0)} &= 3 \geq 3 = 3 - 0 \\ c_{(2,0,2)} &= 1 \geq 0 = 2 - 2 \\ c_{(0,1,3)} &= -2 \geq -3 = 0 - 3. \end{aligned}$$

It follows that $F = (1\odot x \oplus 0\odot y \oplus 1\odot z)^4$.

Example 4.5. By theorem 4.2 holds $F := 0\odot x^4 \oplus 3\odot x^3 y \oplus 3\odot x^2 y^2 \oplus 1\odot xy^3 \oplus 0\odot y^4 = 0\odot x^4 \oplus 0\odot y^4$, so $F \in B_{2d}$. However, the first representation of F is not in the zero set of the tropicalized Catalecticant

$$\begin{aligned} C_4(0, 1, 3, 3, 0) &= \min(0 + 3 + 0, 0 + 2 \cdot 3, 2 \cdot 1 + 0, 1 + 3 + 3, 3 \cdot 3) \\ &= \min(3, 6, 2, 7, 9) \end{aligned}$$

while the second one is:

$$\begin{aligned} C_4(0, \infty, \infty, \infty, 0) &= \min(0 + \infty + 0, 0 + 2 \cdot \infty, 2 \cdot \infty + 0, \infty + \infty + \infty, 3 \cdot \infty) \\ &= \min(\infty, \infty, \infty, \infty, \infty, \infty). \end{aligned}$$

Remark 4.6. If we take all coefficients of a polynomial $F \in \mathbb{T}^d[x_1, ..., x_n]$ to be greater or equal to zero and the coefficients c_{de_k} to not be infinity, we get a geometrical interpretation of proposition 4.1 and theorem 4.2.
Consider the n points $(de_1, c_{de_1}), ..., (de_n, c_{de_n}) \in \mathbb{R}_x^n \times \mathbb{R}_y$. These points are obviously linearly independent and hence define a hyperplane H in $\mathbb{R}_x^n \times \mathbb{R}_y$. Basic linear algebra gives us the defining equation of H as $\frac{c_{de_1}}{d} x_1 + ... + \frac{c_{de_n}}{d} x_n - y = 0$, see calculation 6.1 in the appendix.
This results in the interpretation that a summand $c_i \odot x^i$ of a tropical polynomial does not contribute to its function if the point $(i_1, ..., i_n, c_i)$ lies above the hyperplane H - in the sense that the ray $(i_1, ..., i_n, c_i) + t(0, ..., 0, 1), 0 \leq t \in \mathbb{R}$ does not intersect with H.

In the previous section, we always excluded the case that d is one. However, now we can deal with that case.

Corollary 4.7. *It holds* $V^{\text{trop}}(\operatorname{trop} C_2) = \overline{\operatorname{trop} A_2} = B_2$.

Proof. Let $F = c_0 \odot y^2 \oplus c_1 \odot xy \oplus c_2 \odot x^2$ be in $V^{\text{trop}}(\operatorname{trop} C_2) = V^{\text{trop}}(c_2 c_0 \oplus c_1^2)$. Then $c_2 + c_0 = 2c_1$ holds. Equivalently, it holds $c_1 = \frac{c_0}{2} + \frac{c_2}{2}$. By theorem 4.2 follows $F \in B_2$, hence $V^{\text{trop}}(\operatorname{trop} C_2) \subset B_2$.
Let $F = 2\mu \odot x^2 \oplus 2\nu y^2$ be in B_2. By theorem 4.2 holds $F = 2\mu \odot x^2 \oplus (\mu + \nu) \odot xy \oplus 2\nu y^2$. In particular $F \in V^{\text{trop}}(\operatorname{trop} C_2)$. □

Example 4.8. We consider the polynomial $f = x^2 + 2(1+t)xy + y^2$. Since $C_2(f) = 1^2 \cdot 1^2 - (t+1)^2 \neq 0$, we see that $f \notin A_2$. Nevertheless, it holds $\operatorname{trop} f = x^2 \oplus xy \oplus y^2 = (x \oplus y)^2$. This means we found a polynomial $f \notin A_2$ for which its tropicalization is still an element of B_2.

4.2 Application on tropicalized Polynomials

In this section, we will apply the results from the previous subsection to specify under which conditions the special property of the Catalecticant stated in theorem 2.5 by Sylvester remains true after tropicalization. The following corollary gives us an equivalent version of the inequalities of theorem 4.2.

Corollary 4.9. *Let $F \in \mathbb{T}^d[x,y]$, c_i be the coefficient of $x^i y^{d-i}$ and $c_0 \leq c_d$. Then F is the d-th power of the tropical linear form $\frac{c_0}{d} \odot y \oplus \frac{c_d}{d} \odot x$ if, and only if, it holds*

$$c_i \geq c_0 + i\delta$$

where $\delta = \frac{c_d - c_0}{d}$ for all $i = 0, ..., d$.

Proof. It holds $c_0 + i\delta = c_0 + i\frac{c_d - c_0}{d} = i\frac{c_d}{d} + (d-i)\frac{c_0}{d}$ and hence the result follows from theorem 4.2. □

Proposition 4.10. *Let $f = \sum_{i=1}^{d} (\mu_i x + \nu_i y)^{2d} \in \mathfrak{C}^{2d}[x,y]$ with $\mu_i, \nu_i \in \mathfrak{C}, i = 1, ..., d$, i.e. $f \in V(C_{2d})$ and $u_i := \operatorname{val} \mu_i$, $v_i := \operatorname{val} \nu_i$.*
Then it holds $\operatorname{trop} f \in B_{2d}$ if

$$\operatorname{val}(\sum_{i=1}^{d} \mu_i^{2d}) = 2d \min_{i \in \{1,...,d\}} \{u_i\} \tag{3}$$

and

$$\operatorname{val}(\sum_{i=1}^{d} \nu_i^{2d}) = 2d \min_{i \in \{1,...,d\}} \{v_i\}. \tag{4}$$

In particular, we get $\operatorname{trop} f = (\operatorname{val}(\sum_{i=1}^{d} \mu_i) \odot x \oplus \operatorname{val}(\sum_{i=1}^{d} \nu_i) \odot y)^{2d}$.

Proof. Let f and μ_i, ν_i, u_i, v_i be as stated above. We calculate the coefficient c_k of $F := \operatorname{trop} f$ belonging to $x^k \odot y^{2d-k}$ as

$$c_k = \operatorname{val}(\sum_{i=1}^{d} \binom{2d}{k} \mu_i^k \nu_i^{2d-k}) \geq \min\{ku_1 + (2d-k)v_1, ..., ku_d + (2d-k)v_d\}.$$

Let equations (3) and (4) be true. It follows that $c_0 = \min_{i \in \{1,...,d\}} \{2dv_i\}$ and $c_{2d} = \min_{i \in \{1,...,d\}} \{2du_i\}$. We compute

$$\delta = \frac{c_{2d} - c_0}{2d} = \min_{i \in \{1,...,d\}} \{u_i\} - \min_{i \in \{1,...,d\}} \{v_i\}.$$

We get

$$\begin{aligned}
c_k =& \operatorname{val}(\sum_{i=1}^{d}\binom{2d}{k}\mu_i^k\nu_i^{2d-k}) \geq \min\{ku_1 + (2d-k)v_1, ..., ku_d + (2d-k)v_d\} \\
\geq& k \min_{i\in\{1,\dots,d\}}\{u_i\} + (2d-k)\min_{i\in\{1,\dots,d\}}\{v_i\} \\
=& k \cdot \left(\min_{i\in\{1,\dots,d\}}\{u_i\} - \min_{i\in\{1,\dots,d\}}\{v_i\}\right) + 2d \min_{i\in\{1,\dots,d\}}\{v_i\} \\
=& k\cdot\delta + c_0.
\end{aligned}$$

Applying corollary 4.9, we are finished. □

By lemma 2.16 these conditions are for example fulfilled if $\operatorname{val}\mu_i \neq \operatorname{val}\mu_j$ and $\operatorname{val}\nu_i \neq \operatorname{val}\nu_j$ for all $i \neq j$.

5 Tropicalization of the Lüroth Invariant

The next invariant for which we will discuss its tropicalization is the Lüroth invariant. In the first subsection we will introduce this invariant and Lüroth quartics, in the second one we will discuss how Lüroth quartics behave under tropicalization.
Again, we will see that the tropicalized invariant does not have the same property as the classical one.

5.1 The Lüroth Invariant

The non-tropical definitions and statements are taken from [OS10] chapter 11. To introduce this invariant, we first need to give some definitions and clarify notation.
We will again work over the field of Puiseux series with coefficients in the complex numbers, $\mathfrak{C}$. The n-dimensional projective space over $\mathfrak{C}$ will be denoted $\mathbb{P}^n$. If the ground field is different to $\mathfrak{C}$, this will be noted in the index, e.g. by $\mathbb{P}^3_{\mathbb{C}}$ we will mean the three-dimensional projective space over the complex numbers.

Definition 5.1 (Complete Pentalateral)**.** A configuration consisting of five lines in $\mathbb{P}^2$, three by three linearly independent, together with the ten double points of their union will be called a *complete pentalateral.*
The ten nodes of their union are called *vertices* of the complete pentalateral.

Definition 5.2 (Lüroth Quartic)**.** A *Lüroth quartic* is a nonsingular quartic which contains the ten vertices of a complete pentalateral.

This property also gives us information about how the defining polynomial can be written.

Proposition 5.3. *Given a complete pentalateral in $\mathbb{P}^2$ defined by lines $l_1, ..., l_5$ with vertex set $\mathbf{P} \subset \mathbb{P}^2$ the quartics containing the vertices $\mathbf{P}$ are of the form*

$$\lambda_5 l_1l_2l_3l_4 + \lambda_4 l_1l_2l_3l_5 + \lambda_3 l_1l_2l_4l_5 + \lambda_2 l_1l_3l_4l_5 + \lambda_1 l_2l_3l_4l_5 \tag{5}$$

where $(\lambda_1 : ... : \lambda_5) \in \mathbb{P}^4$.

These quartics define a hypersurface in $\mathbb{P}(\mathfrak{C}^4[x, y, z])$.

Definition 5.4 (Lüroth Invariant)**.** The polynomial defining this hypersurface is called the *Lüroth invariant* and will be denoted $\mathfrak{L}$.

As already implicitly stated, $\mathfrak{L}$ is an invariant.
By definition holds for a polynomial $f \in \mathfrak{C}^4[x, y, z]$ that $\mathfrak{L}$ vanishes if, and only if, $V(f)$ contains the ten vertices of a complete pentalateral. However, since the zero set of $\mathfrak{L}$ is the closure of the set of Lüroth quartics, the vanishing of $\mathfrak{L}$ does not imply that f is a Lüroth quartic. For example if we take $\lambda = (1 : 0 : 0 : 0 : 0)$ the curve will be singular.

Remark 5.5. Although the existence of the Lüroth invariant was known for a long time - since 1868 - and since 1919 its degree, an explicit expression was not. Only in 2013, an explicit expression was computed in [BLRS13] using several hours of CPU time.

5.2 Tropicalizing the Lüroth Invariant

Analogously to the projective space $\mathbb{P}^n_{\mathbb{K}}$ over a field $\mathbb{K}$, we define the tropical projective space $\mathbb{TP}^n$. This definition is taken from [MR].

Definition 5.6 (Tropical projective Space)**.** The *tropical projective space* $\mathbb{TP}^n$ is defined as

$$\mathbb{TP}^n := (\mathbb{T}^{n+1} \backslash \{\infty\})/\mathbb{R}(1, ..., 1)$$

where ∞ denotes the point $(\infty, ..., \infty)$.
We write $P = (p_0 : ... : p_n)$ for a point in $\mathbb{TP}^n$.

Of course Kapranov's theorem also holds for projective varieties.

Theorem 5.7. *For* $f \in \mathfrak{C}^d[x_0, ..., x_n]$ *the tropical hyperplane* $V^{\text{trop}}(\operatorname{trop} f) := \{P = (p_0 : ... : p_n) \in \mathbb{TP}^n \mid \textit{minimum of} \operatorname{trop} f \textit{ attained twice for } P\}$ *in* $\mathbb{TP}^n$ *and the topological closure of* $\{(\operatorname{val} y_0 : ... : \operatorname{val} y_n) \mid y \in V(f)\}$ *in* $\mathbb{TP}^n$ *coincide.*

We directly see that the vertices of a pentalateral remain in the zero set of the tropicalized polynomial and that the image of these points is still in the intersection of the tropicalization of the original lines.

Proposition 5.8. *Let* $f \in V(\mathfrak{L})$ *and* $\mathbf{P} \subset V(f)$ *be the vertex set of a complete pentalateral. Then* $\operatorname{val}(\mathbf{P}) \subset V^{\text{trop}}(\operatorname{trop} f))$ *holds.*

Proof. This follows from the fact that $\operatorname{val}(\mathbf{P}) \subset \operatorname{val} V(f) \subset V^{\text{trop}}(\operatorname{trop} f)$. □

Proposition 5.9. *Let* l_1 *and* l_2 *be two lines in* $\mathbb{P}^2$ *and* $P \in V(l_1) \cap V(l_2)$. *Then* $\operatorname{val} P \in V^{\text{trop}}(\operatorname{trop} l_1) \cap V^{\text{trop}}(\operatorname{trop} l_2)$.

Proof. This follows directly from Kapranov's theorem. □

However, in general a complete pentalateral degenerates under val.

Example 5.10. Consider any complete pentalateral in $\mathbb{P}^2_{\mathbb{C}}$ such that all coefficients of the defining lines are unequal to zero, e.g.

$$\begin{aligned} l_1 &: 2x + 1y + 1z \\ l_2 &: 1x + 2y + 1z \\ l_3 &: 1x + 1y + 2z \\ l_4 &: 1x + 1y + 1z \\ l_5 &: 1x + 2y + 3z. \end{aligned}$$

We now interpret these lines and the complete pentalateral over $\mathfrak{C} \supset \mathbb{C}$. It holds $\operatorname{trop} l_i = 0 \odot x \oplus 0 \odot y \oplus 0 \odot z$ for $i = 1, ..., 5$. In conclusion, the pentalateral degenerated to a single tropical line.

It is known that two different tropical lines in $\mathbb{TP}^2$ intersect in exactly one or infinitely many points [BS14]. In the second case, the intersection points are all of the form $(p_1 + \varepsilon : p_2 : p_3), (p_1 : p_2 + \varepsilon : p_3)$ or $(p_1 : p_2 : p_3 + \varepsilon)$ for an arbitrary intersection point $P = (p_1 : p_2 : p_3)$, $\varepsilon \in \mathbb{R}$. In particular, this means that the minimum of the two intersecting lines must be attained by the same entries for the intersection points if the lines have more than one intersection point.
The next definition follows [MS15]. We omit the prove of its well-definedness.

Definition 5.11 (Stable Intersection)**.** Let C and D be two tropical curves. Shift C by a vector v such that the intersection of $C + v$ and D is finite. Then the points in $\lim_{\varepsilon \to 0}(C + \varepsilon v) \cap D$ are called the *stable intersection points* of C and D.

In the case of lines in $\mathbb{TP}^2$ intersecting in an infinite number of points, the stable intersection point is the point for which for one of the lines it lies on the minimum is attained three times. It holds that two tropical lines have exactly one (stable) intersection point.
Similarly to the definition of a complete pentalateral above, we define a complete tropical pentalateral.

Definition 5.12 (Complete tropical Pentalateral)**.** A configuration consisting of five tropical lines in $\mathbb{TP}^2$ which have ten different stable intersection points is called a *complete tropical pentalateral.*
These ten points are called *vertices* of the complete tropical pentalateral.

The next proposition proves one direction of the connection between a quartic containing the vertices of a complete pentalateral and a special representation of its defining polynomial for the tropical case.

Proposition 5.13. *Let $F \in \mathbb{T}^4[X, Y, Z]$. $V^{\text{trop}}(F)$ contains the intersection points of any pair of the lines $L_1, ..., L_5$ if F can be written as*

$$\lambda_5 \odot L_1 L_2 L_3 L_4 \oplus \lambda_4 \odot L_1 L_2 L_3 L_5 \oplus \lambda_3 \odot L_1 L_2 L_4 L_5 \oplus \lambda_2 \odot L_1 L_3 L_4 L_5 \oplus \lambda_1 \odot L_2 L_3 L_4 L_5$$

where $\lambda \in \mathbb{TP}^4$.

In particular, if the lines define a complete tropical pentalateral, $V^{\text{trop}}(F)$ contains its vertices.

Proof. Since for any chosen pair L_i, L_j of lines in each summand of F at least one of the two linear forms occurs, a point which is in the zero set of both lines is also in the zero set of F. □

The following example shows that the implication stated in proposition 5.13 is not an equivalence, even if the tropicalization of a complete pentalateral remains complete.
In conclusion, the vanishing of the tropicalized Lüroth invariant does not give us information about a special representation of F.

Example 5.14. We consider the following five lines:

$$\begin{aligned} l_1 &: tx + 1y + t^4 z \\ l_2 &: 1x + t^6 y + t^2 z \\ l_3 &: 1x + ty + 1z \\ l_4 &: 1x + 1y + tz \\ l_5 &: tx + t^3 y + 1z. \end{aligned}$$

These five lines define a complete pentalateral, see calculation 6.2 in the appendix. We denote the vertices of this pentalateral by $\mathbf{P}$.
Define $f = l_1l_2l_3l_4 + 2l_1l_2l_3l_5 + (-1)l_1l_2l_4l_5 + (-1)l_1l_3l_4l_5 + (-1)l_2l_3l_4l_5$. Then $\mathbf{P} \subset V(f)$ or equivalently $f \in V(\mathfrak{L})$. Using the computer algebra system SINGULAR [DGPS20], we can compute that f is a Lüroth quartic, i.e. nonsingular, see code 6.4 and 6.5. Expanding the term gives us that the coefficient to x^4 is $t + 2t^2 - t^2 - t^2 - t = 0$. Moreover, we get the following projective tropical lines by tropicalizing the lines $l_1, ..., l_5$:

$$\begin{aligned} L_1 &: 1{\odot}x \oplus 0{\odot}y \oplus 4{\odot}z \\ L_2 &: 0{\odot}x \oplus 6{\odot}y \oplus 2{\odot}z \\ L_3 &: 0{\odot}x \oplus 1{\odot}y \oplus 0{\odot}z \\ L_4 &: 0{\odot}x \oplus 0{\odot}y \oplus 1{\odot}z \\ L_5 &: 1{\odot}x \oplus 3{\odot}y \oplus 0{\odot}z \end{aligned}$$

These lines define a complete tropical pentalateral, see figure 2. The formal proof can be found in the appendix, calculation 6.3. By Proposition 5.8 holds $\operatorname{val}(\mathbf{P}) \subset V^{\text{trop}}(\operatorname{trop} f)$. The coefficient of $\operatorname{trop} f$ belonging to x^4 is $\operatorname{val} 0 = \infty$. But since for all L_i the x-coefficient is unequal to ∞, $\operatorname{trop} f$ can not be of the form $\lambda_5 L_1L_2L_3L_4 \oplus \lambda_4{\odot}L_1L_2L_3L_5 \oplus \lambda_3{\odot}L_1L_2L_4L_5 \oplus \lambda_2{\odot}L_1L_3L_4L_5 \oplus \lambda_1{\odot}L_2L_3L_4L_5$ with $\lambda \in \mathbb{TP}^4$.

Remark 5.15. Furthermore, $\operatorname{trop} f$ can not be of the form

$$\lambda_5 J_1J_2J_3J_4 \oplus \lambda_4{\odot}J_1J_2J_3J_5 \oplus \lambda_3{\odot}J_1J_2J_4J_5 \oplus \lambda_2{\odot}J_1J_3J_4J_5 \oplus \lambda_1{\odot}J_2J_3J_4J_5$$

with $\lambda \in \mathbb{TP}^4$ for any other complete tropical pentalateral defined by lines $J_1, ..., J_5$ either. This is due to the fact that all J_i would have to have ∞ as their x-coefficient. But then all these lines have the stable intersection point $(0 : \infty : \infty)$ and hence can not define a complete pentalateral.

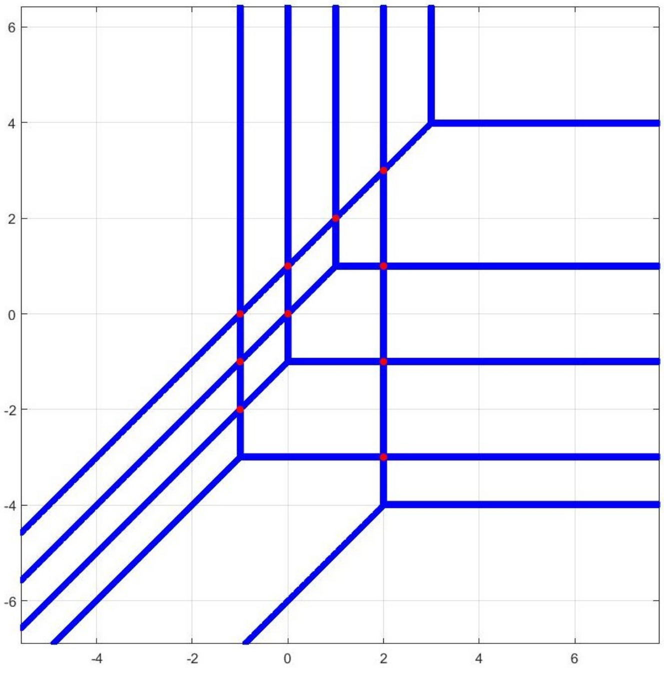

Figure 2: The tropical lines $L_1, ..., L_5$ and the vertices of the complete tropical pentalateral (red dots) under the map $z = 0$

Remark 5.16. Using SINGULAR [DGPS20], for the code see 6.4, we can expand the polynomial defined above and get

$$
\begin{aligned}
f =&(t^8 - 2t^2 + 1)x^3y \\
&+(t^{10} + t^9 - t^8 + t^7 + t^6 + t^5 - 3t^4 - 2t^3 + t^2 - t + 1)x^2y^2 \\
&+(2t^{11} - 2t^{10} + 2t^8 + t^6 - t^5 - 2t^3 - t^2 + t)xy^3 \\
&+(t^{10} - t^9 + t^7 - t^4)y^4 \\
&+(2t^4 - 2t^3 + t^2 - 1)x^3z \\
&+(t^{11} + t^{10}t^9 + 2t^8 + t^7 + t^6 - 2t^5 - t^4 - t^2 - t)x^2yz \\
&+(t^{13} + 2t^{12} - t^{11} + 2t^{10} - t^9 + 2t^8 + t^7 - 3t^6 - t^4 - t^3 + t^2 - t - 1)xy^2z \\
&+(2t^{14} - t^{13} - t^{10} + t^9 + t^7 + t^6 - 2t^5 - t)y^3z \\
&+(t^7 - t^6 + t^5 + 3t^4 - 3t^2 - 1)x^2z^2 \\
&+(-t^{12} + 3t^{11} + 2t^{10} + t^9 - t^8 + 2t^6 - 2t^5 - t^4 - t^2 - 2t)xyz^2 \\
&+(-t^{14} + 2t^{13} + t^{12} + 2t^{11} + t^{10} - 2t^9 - t^8 - t^7 + t^3 - t^2 - 1)y^2z^2 \\
&+(-t^8 + 3t^7 + t^6 - t^5 + t^3 - 2t^2 - t)xz^3 \\
&+(t^{10} + 2t^9 + t^7 - 2t^6 - 2t^4 + t^2 - t)yz^3 \\
&+(2t^6 - t^5 - t^3)z^4.
\end{aligned}
$$

This allows us to compute the tropicalization of f. It holds

$$\begin{aligned}\operatorname{trop} f =& 0\odot x^3y \oplus 0\odot x^2y^2 \oplus 1\odot xy^3 \oplus 4\odot y^4 \oplus 0\odot x^3z \\ &\oplus 1\odot x^2yz \oplus 0\odot xy^2z \oplus 1\odot y^3z \oplus 0\odot x^2z^2 \oplus 1\odot xyz^2 \\ &\oplus 0\odot y^2z^2 \oplus 1\odot xz^3 \oplus 1\odot yz^3 \oplus 3\odot z^4.\end{aligned}$$

A visualization can be seen in figure 3.

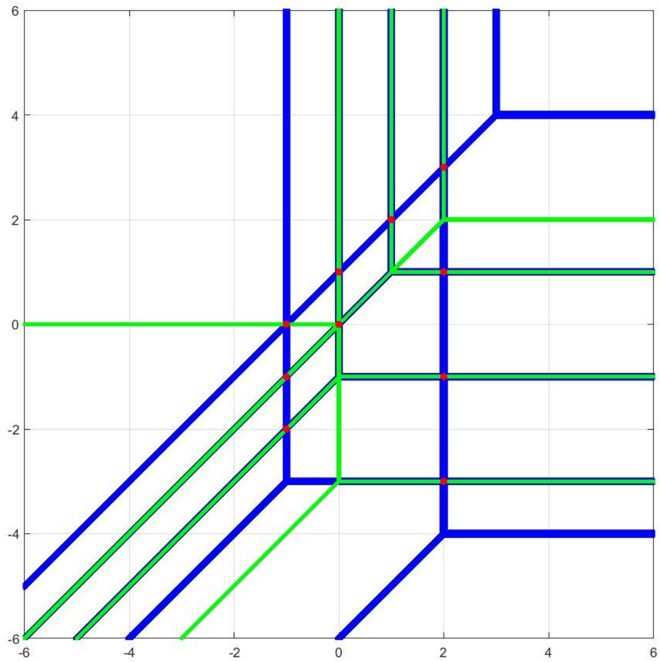

Figure 3: The tropical lines $L_1, ..., L_5$ (blue), the vertices of the complete tropical pentalateral (red) and $V^{\text{trop}}(\operatorname{trop} f)$ (green) under the map $z = 0$

6 Appendix

Calculation 6.1 (For remark 4.6)**.** We compute the equation defining the hyperplane $H = \langle (de_1, c_{de_1}), ..., (de_n, c_{de_n}) \rangle$ in $\mathbb{R}^n_x \times \mathbb{R}_y$. The algorithm is basic linear algebra.
Define the Matrix $B \in \mathrm{Mat}(n \times n+1, \mathfrak{C})$ as

$$B := \begin{pmatrix} d & 0 & \cdots & \cdots & 0 & c_{de_1} \\ 0 & d & 0 & \cdots & 0 & c_{de_2} \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & d & c_{de_n} \end{pmatrix}.$$

If we divide each row of B by d, we see that the solution of the linear equation system $Bx = 0$ is $(\frac{c_{de_1}}{d}, ..., \frac{c_{de_n}}{d}, -1)^t$. Hence, H is defined by the equation $\frac{c_{de_1}}{d}x_1 + ... + \frac{c_{de_n}}{d}x_n - y = 0$.

Calculation 6.2 (For example 5.14)**.** We prove that the five lines

$$\begin{aligned} l_1 &: tx + 1y + t^4 z \\ l_2 &: 1x + t^6 y + t^2 z \\ l_3 &: 1x + ty + 1z \\ l_4 &: 1x + 1y + tz \\ l_5 &: tx + t^3 y + 1z \end{aligned}$$

define a complete pentalateral. To do so we need to show that three different lines l_i, l_j, l_k are always linearly independent. Equivalently, we need to compute the following $\binom{5}{3} = 10$ determinants. These computations were done using the calculator at https://matrixcalc.org/de/. For $(i, j, k) =$

- $(1, 2, 3)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & t^6 & t^2 \\ 1 & t & 1 \end{vmatrix} = -t^{10} + t^7 + t^5 - t^4 + t^2 - 1 \neq 0.$

- $(1, 2, 4)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & t^6 & t^2 \\ 1 & 1 & t \end{vmatrix} = -t^{10} + t^8 + t^4 - t^3 + t^2 - t \neq 0.$

- $(1, 2, 5)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & t^6 & t^2 \\ t & t^3 & 1 \end{vmatrix} = -t^{11} + 2t^7 - t^6 + t^3 - 1 \neq 0.$

- $(1, 3, 4)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = -t^5 + t^4 + t^3 - 2t + 1 \neq 0.$

- $(1, 3, 5)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & t & 1 \\ t & t^3 & t \end{vmatrix} = t^7 - t^6 - t^4 + t^3 \neq 0.$

- $(1,4,5)$ holds: $\begin{vmatrix} t & 1 & t^4 \\ 1 & 1 & t \\ t & t^3 & 1 \end{vmatrix} = t^7 - 2t^5 + t^2 + t - 1 \neq 0.$

- $(2,3,4)$ holds: $\begin{vmatrix} 1 & t^6 & t^2 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = -t^7 + t^6 - t^3 + 2t^2 - 1 \neq 0.$

- $(2,3,5)$ holds: $\begin{vmatrix} 1 & t^6 & t^2 \\ 1 & t & 1 \\ t & t^3 & 1 \end{vmatrix} = t^7 - t^6 + t^5 - t^4 - t^3 + t \neq 0.$

- $(2,4,5)$ holds: $\begin{vmatrix} 1 & t^6 & t^2 \\ 1 & 1 & t \\ t & t^3 & 1 \end{vmatrix} = t^8 - t^6 + t^5 - t^4 - t^3 + 1 \neq 0.$

- $(3,4,5)$ holds: $\begin{vmatrix} 1 & t & 1 \\ 1 & 1 & t \\ t & t^3 & 1 \end{vmatrix} = -t^4 + 2t^3 - 2t + 1 \neq 0.$

We see that all determinants are unequal to zero and hence each triple of lines l_i, l_j, l_k is linearly independent.

Calculation 6.3 (For example 5.14)**.** We show that the five tropical lines

$$\begin{aligned} L_1 &: 1 \odot x \oplus 0 \odot y \oplus 4 \odot z \\ L_2 &: 0 \odot x \oplus 6 \odot y \oplus 2 \odot z \\ L_3 &: 0 \odot x \oplus 1 \odot y \oplus 0 \odot z \\ L_4 &: 0 \odot x \oplus 0 \odot y \oplus 1 \odot z \\ L_5 &: 1 \odot x \oplus 3 \odot y \oplus 0 \odot z \end{aligned}$$

define a complete tropical pentalateral by giving the ten intersection points of the lines. The intersection point of L_i and L_j for $(i,j) =$

- $(1,2)$ is $P = (2:3:0)$: $L_1(P) = \min(3,3,4)$ and $L_2(P) = \min(2,9,2)$.
- $(1,3)$ is $P = (0:1:0)$: $L_1(P) = \min(1,1,4)$ and $L_3(P) = \min(0,2,0)$.
- $(1,4)$ is $P = (1:2:0)$: $L_1(P) = \min(2,2,4)$ and $L_4(P) = \min(1,2,1)$.
- $(1,5)$ is $P = (-1:0:0)$: $L_1(P) = \min(0,0,4)$ and $L_5(P) = \min(0,3,0)$.
- $(2,3)$ is $P = (2:-1:0)$: $L_2(P) = \min(2,5,2)$ and $L_3(P) = \min(2,0,0)$.
- $(2,4)$ is $P = (2:1:0)$: $L_2(P) = \min(2,7,2)$ and $L_4(P) = \min(2,1,1)$.
- $(2,5)$ is $P = (2:-3:0)$: $L_2(P) = \min(2,3,2)$ and $L_5(P) = \min(3,0,0)$.
- $(3,4)$ is $P = (0:0:0)$: $L_3(P) = \min(0,1,0)$ and $L_4(P) = \min(0,0,1)$.
- $(3,5)$ is $P = (-1:-2:0)$: $L_3(P) = \min(-1,-1,0)$ and $L_5(P) = \min(0,1,0)$.
- $(4,5)$ is $P = (-1:-1:0)$: $L_4(P) = \min(-1,-1,1)$ and $L_5(P) = \min(0,2,0)$.

Since the minima above are always attained by different pairs for each point, the listed points are the only intersection points.
Hence we have ten intersection points and a complete tropical pentalateral.

Code 6.4 (For examples 5.14 and 5.16)**.** This is the code used in SINGULAR to compute f from example 5.14. It is straight forward, just a simple computation. Since only polynomials with coefficients in $\mathbb{C}[t]$ (the ring `r`) occur, the computation was done over $\mathbb{C}[t][x, y, z]$ (the ring `q`).

```
> ring r = complex,t,dp;
> ring q = r,(x,y,z),dp;
> poly le = (tx+y+t4z);
> poly lz = (x+t6y+t2z);
> poly ld = (x+ty+z);
> poly lv = (x+y+tz);
> poly lf = tx+t3y+z;
> poly f = le*lz*ld*lv + 2*le*lz*ld*lf - le*lz*lv*lf
. - le*ld*lv*lf - lz*ld*lv*lf;
> print(f);
```

To avoid the extreme long term, the last output was omitted.

Code 6.5 (For example 5.14)**.** We prove that the polynomial f in example 5.14 is nonsingular.
To do this, we show that under the maps $x = 1$, $y = 1$ and $z = 1$ the ideal of f and its two derivations contains 1. All calculations were done over $\mathbb{Q}(t)$. The derivations and dehomogenisations were done by hand.
For the map $x = 1$:

```
> ring r = (0,t), (y,z), dp;
> poly f = (t8-2*t2+1)*y+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*y2+(2*t11-2*t10+2
*t8+t6-t5-2*t3-t2+t)*y3+(t10-t9+t7-t4)*y4+(2*t4-2*t3+t2-1)*z+(t11+t10-t9+2*t8+t7
+t6-2*t5-t4-t2-t)*yz+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*y2z+(2*t
14-t13-t10+t9+t7+t6-2*t5-t)*y3z+(t7-t6+t5+3*t4-3*t2-1)*z2+(-t12+3*t11+2*t10+t9-t
8+2*t6-2*t5-t4-t2-2*t)*yz2+(-t14+2*t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*y2z2+(-
t8+3*t7+t6-t5+t3-2*t2-t)*z3+(t10+2*t9+t7-2*t6-2*t4+t2-t)*yz3+(2*t6-t5-t3)*z4;
> poly yf = (t8-2*t2+1)+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*2*y+(2*t11-2*t10+2
*t8+t6-t5-2*t3-t2+t)*3*y2+(t10-t9+t7-t4)*4*y3+(t11+t10-t9+2*t8+t7+t6-2*t5-t4-t2-
t)*z+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*2*yz+(2*t14-t13-t10+t9+t
7+t6-2*t5-t)*3*y2z+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*z2+(-t14+2*t13+t
12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*2*yz2+(t10+2*t9+t7-2*t6-2*t4+t2-t)*z3;
> poly zf = (2*t4-2*t3+t2-1)+(t11+t10-t9+2*t8+t7+t6-2*t5-t4-t2-t)*y+(t13+2*t12-t
11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*y2+(2*t14-t13-t10+t9+t7+t6-2*t5-t)*y3+(t7
-t6+t5+3*t4-3*t2-1)*2*z+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*2*yz+(-t14+
2*t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*2*y2z+(-t8+3*t7+t6-t5+t3-2*t2-t)*3*z2+(t
10+2*t9+t7-2*t6-2*t4+t2-t)*3*yz2+(2*t6-t5-t3)*4*z3;
> ideal I = f, yf,zf;
> ideal J = std(I);
> print(J);
1
```

For the map $y = 1$:

```
> ring r = (0,t), (x,z), dp;
> poly f = (t8-2*t2+1)*x3+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*x2+(2*t11-2*t10+
2*t8+t6-t5-2*t3-t2+t)*x+(t10-t9+t7-t4)+(2*t4-2*t3+t2-1)*x3z+(t11+t10-t9+2*t8+t7+
t6-2*t5-t4-t2-t)*x2z+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*xz+(2*t1
4-t13-t10+t9+t7+t6-2*t5-t)*z+(t7-t6+t5+3*t4-3*t2-1)*x2z2+(-t12+3*t11+2*t10+t9-t8
+2*t6-2*t5-t4-t2-2*t)*xz2+(-t14+2*t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*z2+(-t8+
3*t7+t6-t5+t3-2*t2-t)*xz3+(t10+2*t9+t7-2*t6-2*t4+t2-t)*z3+(2*t6-t5-t3)*z4;
> poly xf = (t8-2*t2+1)*3*x2+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*2*x+(2*t11-2*
t10+2*t8+t6-t5-2*t3-t2+t)+(2*t4-2*t3+t2-1)*3*x2z+(t11+t10-t9+2*t8+t7+t6-2*t5-t4-t2
-t)*2*xz+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*z+(t7-t6+t5+3*t4-3*t
2-1)*2*xz2+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*z2+(-t8+3*t7+t6-t5+t3-2*
t2-t)*z3;
> poly zf = (2*t4-2*t3+t2-1)*x3+(t11+t10-t9+2*t8+t7+t6-2*t5-t4-t2-t)*x2+(t13+2*t
12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*x+(2*t14-t13-t10+t9+t7+t6-2*t5-t)+(t7
-t6+t5+3*t4-3*t2-1)*2*x2z+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*2*xz+(-t1
4+2*t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*2*z+(-t8+3*t7+t6-t5+t3-2*t2-t)*3*xz2+(
t10+2*t9+t7-2*t6-2*t4+t2-t)*3*z2+(2*t6-t5-t3)*4*z3;
> ideal I = f,xf,zf;
> ideal J = std(I);
> print(J);
1
```

For the map $z = 1$:

```
>ring r = (0,t), (x,y), dp;
> poly f = (t8-2*t2+1)*x3y+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*x2y2+(2*t11-2*t
10+2*t8+t6-t5-2*t3-t2+t)*xy3+(t10-t9+t7-t4)*y4+(2*t4-2*t3+t2-1)*x3+(t11+t10-t9+2
*t8+t7+t6-2*t5-t4-t2-t)*x2y+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*x
y2+(2*t14-t13-t10+t9+t7+t6-2*t5-t)*y3+(t7-t6+t5+3*t4-3*t2-1)*x2+(-t12+3*t11+2*t1
0+t9-t8+2*t6-2*t5-t4-t2-2*t)*xy+(-t14+2*t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*y2
+(-t8+3*t7+t6-t5+t3-2*t2-t)*x+(t10+2*t9+t7-2*t6-2*t4+t2-t)*y+(2*t6-t5-t3);
> poly xf =(t8-2*t2+1)*3*x2y+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*2*xy2+(2*t11-
2*t10+2*t8+t6-t5-2*t3-t2+t)*y3+(2*t4-2*t3+t2-1)*3*x2+(t11+t10-t9+2*t8+t7+t6-2*t5-t
4-t2-t)*2*xy+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*y2+(t7-t6+t5+3*t
4-3*t2-1)*2*x+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*y+(-t8+3*t7+t6-t5+t3-
2*t2-t);
> poly yf = (t8-2*t2+1)*x3+(t10+t9-t8+t7+t6+t5-3*t4-2*t3+t2-t+1)*2*x2y+(2*t11-2*
t10+2*t8+t6-t5-2*t3-t2+t)*3*xy2+(t10-t9+t7-t4)*4*y3+(t11+t10-t9+2*t8+t7+t6-2*t5-
t4-t2-t)*x2+(t13+2*t12-t11+2*t10-t9+2*t8+t7-3*t6-t4-t3+t2-t-1)*2*xy+(2*t14-t13-t
10+t9+t7+t6-2*t5-t)*3*y2+(-t12+3*t11+2*t10+t9-t8+2*t6-2*t5-t4-t2-2*t)*x+(-t14+2*
t13+t12+2*t11+t10-2*t9-t8-t7+t3-t2-1)*2*y+(t10+2*t9+t7-2*t6-2*t4+t2-t);
> ideal I = f,xf,yf;
> ideal J = std(I);
> print(J);
1
```

Code 6.6. The following code is a self-written MATLAB script that can plot $V^{\text{trop}}(F)$ for any tropical polynomial $F \in \mathbb{T}[x, y]$. It is the basis-code used for all plots occurring in this text.
The course of action is straight forward. For a large number of points, the algorithm does the following: It computes the entries in the minimum of F

for the point. Then it checks whether the minimum is attained twice. If it is attained twice, the point get saved. In the end, all saved points get colored. The code printed here is for the tropical polynomial $F = 2{\odot}x^2 \oplus 0{\odot}x \oplus 0{\odot}y^2 \oplus 1$.

```
%This script plots the zero-set of a bivariate tropical polynomial F

%Input:    Highest occuring exponent of a variable in a tropical
%              polynomial F
%          Coefficients of F
%          Plotting area
%Output:   A Plot of the tropical variety of F

% As an example filled out for F = 2x^2 + 0x + 0y^2 + 1

%% Infinity
infty = 100000000000000; %Auxiliary-infinity

%% Definition of F
d = 2;                   %Highest occuring exponent of a variable
A = infty * ones(d+1,d+1); %Initialization of F

%A(i+1,j+1) is the coefficient that belongs to x^iy^j

A(3,1) = 2;
A(2,1) = 0;
A(1,3) = 0;
A(1,1) = 1;

%% Plottingsettings
xres = 1003;  %Resulotion x-axis
yres = 1003;  %Resolution y-axis
%Square recommended
a_x = -5;     %Plotting-range x-axis left
b_x = 5;      %Plotting-range x-axis right
a_y = -5;     %Plotting-range y-axis left
b_y = 5;      %Plotting-range y-axis right

eps = 0.01; %Epsilon: Start with sth large (e.g. 0.5). Smaller eps gives
%more slim plot but you may lose some lines

%% Calculations
x = linspace(a_x,b_x,xres); %x-values
y = linspace(a_y,b_y,yres); %y-values
X = zeros(xres*yres,1);  %Initialisation
Y = zeros(xres*yres,1);  %Initialisation

%Computes whether a point lies in V^{trop}(F) or not
k = 1;
for i=1:xres
    for j=1:yres
        v = tropPolySet(x(i),y(j),A);
        if lowermin(v,eps) ~= uppermin(v,eps)
            X(k) = x(i);
            Y(k) = y(j);
            k = k + 1;
        end
    end
end
[X,Y] = nozeros(X,Y); %Removes the (0,0) pairs in [X,Y]. Else the zero
                      %point is always included

%% Plotting
```

```
plot(X,Y,'.');                      %Plots the points calculated above
axis([a_x b_x a_y b_y])             %Defines the axis
ratio = (b_x-a_x)/(b_y-a_y);
pbaspect([ratio 1 1])               %Rescales the plot such that the
                                    %lines are (almost) not distorted
set (gcf,'Position', [10 10 900 900]) %Defines how the window containing
                                    %the plot is represented

%% Needed functions %%
%--------------------%

function [v] = tropPolySet(x,y,A)
%TROPPOLYSET: Gives the entries in the minimum of the tropical polynomial
%            defined by A for the point (x,y) as a vector
%   Input:   Two values x and y
%            A matrix A that defines a tropical polynomial F
%   Output:  The vector in the minimum of F

%% Initialisations
d = length(A)-1;      %Highest occuring exponent of a variable in F
v = zeros((d+1)^2,1);

%% Computation
    for i=1:(d+1)
        for j=1:(d+1)
            v((i-1)*(d+1)+j) = (i-1)*y + (j-1)*x + A(i,j);
        end
    end
end

function [w,v] = nozeros(x,y)
%NOZEROS Deletes all (0,0) pairs
%   Works only for two vectors of equal length
%   Input: Two vectors
%   Output: Two vectors

%% Initialisations
n = length(x);
m = length(y);

assert(n==m);
W = zeros(n,1);
V = zeros(n,1);
k = 0;

%% Computations
for i=1:n
    if (x(i)~=0) || (y(i)~=0) %If both entries are not zero, copy them
        k = k+1;
        W(k) = x(i);
        V(k) = y(i);
    end
end
w = W(1:k);
v = V(1:k);
end

function [lminind] = lowermin(v,eps)
%LOWERMIN: Gives the minimal index belonging to the minimal entry in v up
%         to esp
%   Used to check if the minimal value in the minimum is attained twice
%   Input: vector v
```

```
%           value eps
%   Output: integer

%% Initialisations
n = length(v);
lmin = v(1);
lminind = 1;

%% Computation
for i=1:1:n
    if v(i) >= lmin + eps
        %Do nothing
    else
        lmin = v(i);
        lminind = i;
    end
end
end

function [uminind] = uppermin(v,eps)
%UPPERMIN: Gives the maximal index belonging to the minimal entry in v up
%          to esp
%   Used to check if the minimal value in the minimum is attained twice
%   Input: vector v
%           value eps
%   Output: integer

%% Initialisations
n = length(v);
umin = v(n);
uminind = n;

%% Computation
for i=n:-1:1
    if v(i) >= umin + eps
        %Do nothing
    else
        umin = v(i);
        uminind = i;
    end
end
end
```

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Eidesstattliche Erklärung

Ich versichere, die vorliegende Arbeit selbstständig verfasst zu haben. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder nicht veröffentlichten Arbeiten anderer entnommen sind, habe ich als entnommen kenntlich gemacht. Sämtliche Quellen und Hilfsmittel, die ich für die Arbeit benutzt habe, sind angegeben. Die Arbeit hat mit gleichem Inhalt bzw. in wesentlichen Teilen noch keiner anderen Prüfungsbehörde vorgelegen.

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