Eberhard Karls Universität Tübingen

Masterarbeit

with maybe more than just a few additional remarks

The k-phase structure

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Contents

1 Introduction

Enumerative geometry is one of the oldest subjects in mathematics: Already the ancient Greeks asked questions like "given three circles in the plane how many different circles are there tangent to these circles?" In the second century B.C., Apollonius could answer this question: The number of circles tangent to three circles in the plane is 8 ([DZ22]).
Problems of this style remained of interest; in the 19th century, Schubert published results like "the number of spheres tangent to 4 general spheres in space is 16" or "the number of conics tangent to 8 quadrics in space is $4,407,296$" ([DZ22]). However, as Schubert's proofs using his theory of characteristics relied on some heuristic arguments, Hilbert asked as the 15th of his famous problems posed at the ICM 1900 in Paris, how these could be done rigorously.
One instance of these enumerative problems is to find the so-called Zeuthen or characteristic numbers: The numbers $N_{d,g}(l)$ of complex plane curves of degree d and genus g tangent to l lines and passing through $k = 3d - 1 + g - l$ general points in the two-dimensional complex projective space $\mathbb{P}^2_{\mathbb{C}}$ (classically, the question is asked for smooth curves, i.e. curves of genus $g = \binom{d-1}{2}$ [Vak99]).
A special case of this problem is to compute the number $N_d := N_{d,0}(0)$ of rational curves of degree d passing through $3d-1$ points in general position. Zeuthen managed to compute those numbers up to $d = 4$ in 1873 ([KLSW23]). In 1994, Kontsevich discovered a recursive formula for N_d ([KM94]).
A more general version, asking for the number $N_{g,d} := N_{g,d}(0)$ of curves with arbitrary genus g through $3d + g - 1$ general points, has been answered in 1996 by Caporaso and Harris via a recursive formula, too ([CH98]).
Nine years later, Mikhalkin showed that these numbers can be computed using tropical geometry ([Mik05]) by proving a correspondence theorem: "The numbers $N_{g,d}$ of complex curves [of genus g] through given points $P_1, ..., P_n$ are the same as the numbers of tropical curves of the same genus and degree through the images of $P_1, ..., P_n$ under the logarithm (resp. valuation) map" counted with some multiplicities ([Gat06]). The formula of Caporaso and Harris can also be proven in tropical geometry — with a much simpler proof. This was done by Gathmann and Markwig ([Gat06],[GM07]).
A natural question to ask next, is what happens if one changes the ground field from the complex numbers $\mathbb{C}$ to the real numbers $\mathbb{R}$. As the real numbers are not algebraically closed, the problem gets more complicated. For example, there can be 8, 10 or 12 rational real cubics passing through 8 real points in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ as Degtyarev and Kharlamov showed ([DK00]). Counted with certain signs, however, Welschinger ([Wel05a],[Wel05b]) showed that these numbers still remain invariant for points in general position. In Mikhalkin's already mentioned paper [Mik05], he showed that these signed counts too can be done via tropical geometry. Noticeably, both of his counts sum over the same set of tropical curves.
More results using signed counts were shown for the question of how many lines there are on a smooth cubic surface in projective space $\mathbb{P}^3_k$. While for $k = \mathbb{C}$ there are always 27 complex lines (Salmon and Cayley in the 19th century [Cay09]), it turns out that for $\mathbb{R}$ there can be either 3, 7, 15 or 27 real lines ([LKW21]). However, there again turned out to be a way to define a signed count that is invariant as Finashin and Kharlamov, and Okonek and Teleman showed ([FK13],[OT14]).

This observation inspired the thought that there might be a more general concept behind changing the base field and getting a different way of counting. Following this thought led to a new way of counting: So-called arithmetic counts were defined. Levine developed a certain invariant arithmetic count of rational curves ([Lev20]), which was later generalized by Kass, Levine, Solomon, and Wickelgren ([KLSW23]). Around the same time, Kass and Wickelgren found an invariant arithmetic count of the lines on a smooth cubic surface ([LKW21]). Shortly after this, Larson and Vogt defined an arithmetic count of bitangents ([LV21]). Moreover, McKean has proven an arithmetic version of Bézout's theorem in [McK21].

For a field k, arithmetic, also called (quadratically) enriched, counts take values in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of quadratic forms over k. This is a direct generalization of the real signed count and the classical complex case in the sense that for $k = \mathbb{C}$ one can recover the classical results over $\mathbb{C}$ by taking the rank, for $k = \mathbb{R}$ by taking the signature.

Recall that Mikhalkin managed in [Mik05] to do both cases, complex and real, simultaneously via tropical geometry, proving a correspondence theorem. This hints at a good compatibility of tropical geometry with arithmetic counts. In [MPS23], Markwig, Payne and Shaw showed that the enriched count of bitangents works in the tropical world, too. A direct generalization of Mikhalkin's correspondence theorem was proven later, as Jaramillo Puentes and Pauli showed that the correspondence is also true for the more general case of arithmetic counting, gaining an arithmetic correspondence theorem ([PP23]). Again, summing over the same set of tropical curves for all fields. In another paper, Jaramillo Puentes and Pauli have proven an enriched version of Bézout's theorem for tropical curves, too ([PP22]).

In this paper, Jaramillo Puentes and Pauli define an enriched intersection multiplicity for enriched tropical hypersurfaces. As the multiplicities in tropical correspondence theorems can often be expressed as such intersection-products, the thought is natural, that there might be a valuable connection to enumerative geometry.

However, this enriched multiplicity is only defined for intersections that cut down to a single point. One motivation for this work was to take first steps towards a possible generalization of this enriched intersection multiplicity to not-0-dimensional intersections.

The idea for enriched tropical hypersurfaces stems from Viro's patchworking ([Vir01]), a tool used to construct real curves with certain topological properties. Viro's patchworking is one of the first instances where tropical geometry occurred. From today's perspective, Viro's patchworking uses quadratical enrichments over the real numbers $\mathbb{R}$, also called distributions of signs.

For a given non-singular tropical hypersurface, [Ren17] defines something called a real phase structure. While a distribution of signs assigns signs on the vertices of the dual of a tropical hypersurface, a real phase is defined directly on the facets of the hypersurface. In [Ren17] it is shown that the two concepts are very closely related.

As already mentioned, the distributions of signs were generalized to arbitrary fields k via quadratical enrichments. A natural question to ask is whether that is possible for the real phase, too: Is it possible to define a k-phase with similar properties? In this text, we answer this question positively. Additionally, we will generalize the k-phase (and hence the real phase) such that singular hyper-

surfaces are not a problem anymore. By a similar procedure as in [Ren17], we will show, that the k-phase structure has the same connection to enrichments as the real phase structure has to distributions of signs.
After establishing the k-phase on hypersurfaces, we will consider intersections of hypersurfaces with k-phases and are able to cover all transversal intersecting cases. On our way to do this, we will look into systems of equations in $({}^{k^*}/_{(k^*)^2})^n$ and develop a theory about their solution sets. Moreover, it will be possible to give a definition of k-phase directly on transversal intersections of arbitrary many tropical hypersurfaces.
Finally, we will see that one of the versions of the enriched versions of Bézout's theorem from [PP22] holds for hypersurfaces with k-phases.
Of course, after generalizing the real phase to general fields, it is an interesting and natural follow-up question which of the results obtained using the real phase can be generalized to other fields, too.

1.1 Acknowledgements

I thank Sabrina Pauli for answers to questions about the structure of ${}^{k^*}/_{(k^*)^2}$ and about her papers and Johannes Rau for answers about the real phase structure.

1.2 Changes

In this text, the points 3.5, 4.44 and the subsubsection 4.6.1 are additions that I made after handing in this text. The rest remained unchanged.

2 Tropical Geometry

We assume that the reader is familiar with the basic concepts of tropical geometry: tropicalization, tropical numbers, varieties and polynomials, the dual subdivisions of tropical hypersurfaces and transversal intersection. We will give a short remark on stable intersection, too. We denote the tropical numbers as $\mathbb{T}$ and use the convention $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with the addition $\oplus$ defined via $a \oplus b = \max\{a, b\}$ and the multiplication $\otimes$ defined via $a \otimes b = a + b$ for $a, b \in \mathbb{T}$. As usual, for $x = (x_1, ..., x_n)$ and $I = (i_1, ..., i_n)$ we define $x^I := x_1^{i_1} \cdots x_n^{i_n}$.

3 Real Phase

In this section, we lay out the real case: We introduce the concepts of distributions of signs and real phase and relate them to each other. We follow [Ren17]. From now on, let C always be a tropical hypersurface in $\mathbb{R}^n$, $n \geq 2$ defined by a tropical polynomial $f = \bigoplus_{I \in \mathbb{N}_0^n} c_I x^I \in \mathbb{T}[x_1, ..., x_n]$ such that each summand with $c_I \neq -\infty$ corresponds to one component of $\mathbb{R}^n \backslash C$. Let Δ denote the dual subdivision with edges Δ_E and vertices Δ_V.
To define a real phase on a tropical hypersurface, we need it to be non-singular.

Definition 3.1 (Non-singular tropical hypersurface)**.** A tropical hypersurface C is called *non-singular* if it is locally a tropical linear space or equivalently, if its dual subdivision is primitive, meaning that each n-dimensional polytope of the subdivision has normalized lattice volume equal to 1.

Now we can give the definition of a real phase. For that, let from now on $\mathcal{F}(C)$ denote the set of facets of a tropical hypersurface C. For every facet $F \in \mathcal{F}(C)$, there exist exactly two monomials $c_I x^I$ and $c_J x^J$ of f that are the maximal ones on F.

Definition 3.2 (corresponding vector)**.** We will call $I - J$ the *corresponding vector* to F.

The order of I and J will not be important. A vector v is in the direction of F if, and only if, $\langle v, I - J \rangle = 0$ for the standard scalar product $\langle \cdot, \cdot \rangle$.

Definition 3.3 (Real phase)**.** A *real phase* on a non-singular tropical hypersurface C in $\mathbb{R}^n$ is the data for every facet $F \in \mathcal{F}(C)$ of 2^{n-1} n-tuples of signs $\varphi_{F,i} = \left(\varphi_{F,i}^1, ..., \varphi_{F,i}^n\right)$, $1 \leq i \leq 2^{n-1}$ satisfying the following properties:

1. If $1 \leq i \leq 2^{n-1}$ and $v = (v_1, ..., v_n)$ is an integer vector in the direction of F, then there exists $1 \leq j \leq 2^{n-1}$ such that $(-1)^{v_k} \varphi_{F,i}^k = \varphi_{F,j}^k$, for $1 \leq k \leq n$.
2. Let H be a codimension 1 face of C. Then for any facet F adjacent to H and any $1 \leq i \leq 2^{n-1}$, there exists a unique face $G \neq F$ adjacent to H and $1 \leq j \leq 2^{n-1}$ such that $\varphi_{G,j} = \varphi_{F,i}$.

We define $\varphi_F := \{\varphi_{F,i} = \left(\varphi_{F,i}^1, ..., \varphi_{F,i}^n\right) \mid 1 \leq i \leq 2^{n-1}\}$. The reason why we need C to be non-singular for this definition will become clear in a later section.

Remark 3.4. If we interpret $+$ and $-$ as 0 and 1 in $\mathbb{Z}_2$, the first property means that the set associated to a facet is an $(n-1)$-dimensional affine hypersurface in $\mathbb{Z}_2^n$.

Remark 3.5. In more recent work, an alternative easier definition of real phase is used (e.g. arXiv:2106.08728v2 definition 2). This definition is equivalent to definition 3.3.
However, if the tropical hypersurface is singular, this definition does *not* coincide with the $\mathbb{R}$-phase defined in this text.
However, I conjecture that the $\mathbb{R}$-phase is closely related to or equivalent to the collection of sets defined in arXiv:2310.08313v1 in definition 4.4. This turns out to be the same as a real phase on a non-singular tropical hypersurface.

To get a better understanding of the definition of real phase, we need to introduce distributions of signs.

Definition 3.6 (Distribution of signs)**.** Let C be a tropical hypersurface in $\mathbb{R}^n$, $n \geq 2$, with dual subdivision Δ. A *distribution of signs* is a map

$$\delta\colon \{\text{vertices of } \Delta\} \to \{+,-\};\ v \mapsto \delta_v.$$

By duality, we could equivalently define δ as a map

$$\{\text{components of } \mathbb{R}^n \backslash C\} \to \{+,-\}.$$

Moreover, it would be equivalent, too, to define δ as a map

$$\delta\colon \{I \in \mathbb{N}_0^n \mid c_I \neq -\infty\} \to \{+,-\};\ I \mapsto \delta_I$$

if $\bigoplus_{I \in \mathbb{N}_0^n} c_I x^I$ is a defining tropical polynomial.

The intuition behind the definitions of real phase and distribution of signs is the following: Every tropical polynomial $F = \bigoplus_I c_I x^I$, $c_I \in \mathbb{Q}$ can be realized as the tropicalization of the polynomial $\sum_I t^{-c_I} x^I$ over the field of Puiseux series $\mathbb{R}\{\{t\}\}$. Of course, this polynomial is not the only one which tropicalizes to f, since (among lots of other information) the sign in front of t^{-c_I} could be changed. By assigning either $+$ or $-$ to each monomial, we fix this sign.
For the polynomial to have a zero, we need the signs of two terms to be opposite. The information stored in the real phase structure are exactly the signs of $x_1, ..., x_n$ needed to have a zero.

Example 3.7. Consider the tropical line in $\mathbb{R}^2$ defined by $x \oplus y \oplus 0$ with the distribution of signs defined by $\delta_{(1,0)} := -$, $\delta_{(0,1)} := -$, and $\delta_{(0,0)} := +$. One element of the preimage of $x \oplus y \oplus 0$ that fits to this distribution of signs is $(-x) + (-y) + 1$. We can now apply the intuition mentioned above to this polynomial to get a corresponding real phase to the distribution of signs. See Figure 1 for a picture of the curve together with its distribution of signs and the real phase.

- φ_{F_1}: For which signs of x and y can $1 + (-y)$ be zero? For y positive and x positive or negative.
- φ_{F_2}: For which signs of x and y can $(-x) + (-y)$ be zero? For x positive and y negative or for x negative and y positive.
- φ_{F_3}: For which signs of x and y can $1 + (-x)$ be zero? For x positive and y positive or negative.

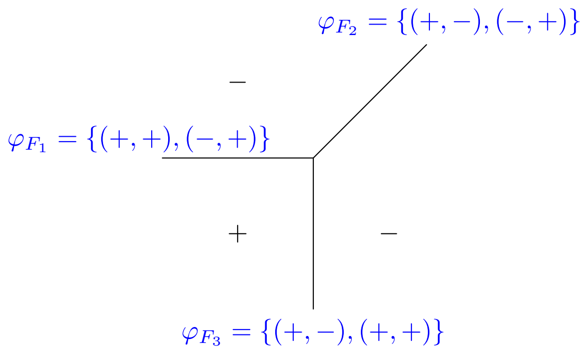


Figure 1: The tropical line with distribution of signs and real phase referenced in Example 3.7.

Example 3.8. In three dimensions, the real phase becomes quite unhandy, it is noted directly in Figure 2. Again, the curve is equipped with a distribution of signs inducing the real phase.

For different distributions of signs we do not always get the same real phase, of course. The next definition states when a real phase is achieved by a certain distribution of signs.

Definition 3.9 (Compatible distribution)**.** A distribution of signs δ at the vertices of Δ is called *compatible with* φ, if for any vertex v of Δ, the following compatibility condition is satisfied:
For any vertex w of Δ adjacent to v, one has $\delta_v \neq \delta_w$ if, and only if, $(+, ..., +) \in \varphi_F$, where F denotes the facet of C dual to the edge connecting v and w.

Using this condition, in [Ren17] the next lemma is proven, which states that there is a two-to-one-correspondence between distributions of signs and real phases.

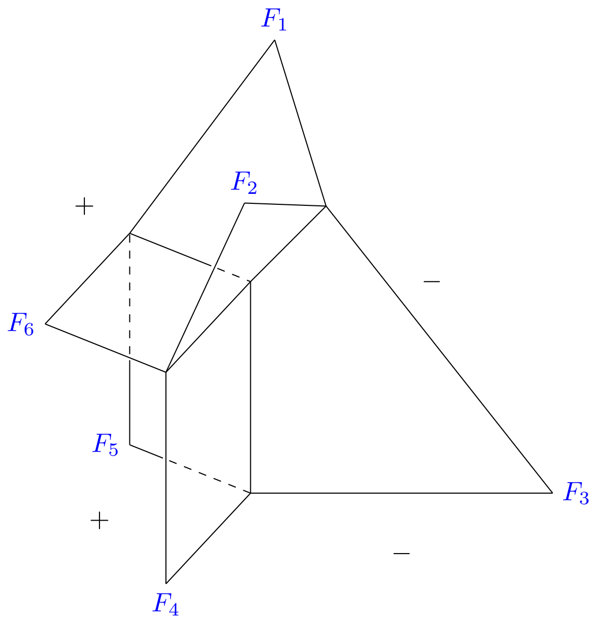
Lemma 3.10. *For any real phase φ on C, there exist exactly two distributions of signs at the vertices of Δ compatible with φ.*
Reciprocally, given any distribution of signs δ at the vertices of Δ, there exists a unique real phase φ on C such that φ is is compatible with δ.

How to get a real phase structure from a distribution of signs was already explained above. To get a fitting distribution of signs from a real phase φ on C works as follows. Choose an arbitrary vertex v of Δ and put an arbitrary sign ϵ at v. Given a vertex of Δ equipped with a sign, define a sign at all adjacent vertices by the compatibility condition from Definition 3.9: $\delta_v \neq \delta_w$ if, and only if, there exists $1 \leq i \leq 2^{n-1}$ such that $\varphi_{F,i} = (+, \cdots, +)$, where F denotes the facet of C dual to the edge connecting v and w.

4 k-phase

4.1 Enriched tropical hypersurfaces

To generalize the concept of a distribution of signs to an arbitrary field k, we replace the set of signs $\{+, -\}$ by the set ${}^{k^*}/_{(k^*)^2}$. For the real case, we get that



- $\varphi_{F_1} = \{(+,+,+),(-,+,+),(+,-,-),(-,-,-)\}$
- $\varphi_{F_2} = \{(+,+,+),(+,-,+),(-,+,-),(-,-,-)\}$
- $\varphi_{F_3} = \{(+,-,+),(+,-,-),(-,+,+),(-,+,-)\}$
- $\varphi_{F_4} = \{(+,+,+),(+,+,-),(+,-,+),(+,-,-)\}$
- $\varphi_{F_5} = \{(+,+,+),(+,+,-),(-,+,+),(-,+,-)\}$
- $\varphi_{F_6} = \{(+,+,-),(-,+,-),(+,-,-),(-,-,-)\}$

Figure 2: Example 3.8 of a tropical hypersurface with distribution of signs and real phase.

${}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2} = \{[-1],[1]\}$. In particular, there are only two classes: One containing the positive real numbers and one containing the negative real numbers. Hence, for $k = \mathbb{R}$ we are in the case of signs again.
Analogously to a distribution of signs, we define an enrichment of a tropical hypersurface ([PP22]).

Definition 4.1 (Enriched tropical hypersurface)**.** An *enrichment* of a tropical hypersurface C is a map

$$\alpha\colon \{\text{vertices of } \Delta\} \to {}^{k^*}/_{(k^*)^2};\ v \mapsto \alpha_v.$$

Together with such a map, we call C an *enriched tropical hypersurface.*

Again, we can equivalently think of α as a map going from the set of components of $\mathbb{R}^n \backslash C$ into ${}^{k^*}/_{(k^*)^2}$. Also, if the defining tropical polynomial is $\bigoplus_{I\in\mathbb{N}_0^n} c_I x^I$, we can equivalently think of it as a map from $\{I \in \mathbb{N}_0^n \mid c_I \neq -\infty\}$

to ${}^{k^*}/_{(k^*)^2}$. We will write α_I for the image of I and call $\bigoplus_{I\in\mathbb{N}_0^n} \alpha_I c_I x^I$ an *enriched tropical polynomial.* Throughout this text, we will switch between these equivalent definitions frequently.

4.2 Definition of k-phase

As we want to follow the same path of thought as for the real phase, we have to think about the following question: If $c_I x^I$ and $c_J x^J$ are the two monomials for which the maximum is attained — what do we need $[x_1], ..., [x_n] \in {}^{k^*}/_{(k^*)^2}$ to be, that there exist some $a_I \in \alpha_I,\ a_J \in \alpha_J,\ y_i \in [x_i],\ i \in \{1, ..., n\}$ such that $a_I y^I + a_J y^J = 0$?
Equivalently, we can ask for the solutions to the equation

$$[a_I][x^I] = [-a_J][x^J] \tag{1}$$

in $({}^{k^*}/_{(k^*)^2})^n$. Hence, before we come to the definition of k-phase, we need to note some properties of ${}^{k^*}/_{(k^*)^2}$.
Obviously, $({}^{k^*}/_{(k^*)^2}, \cdot)$ with multiplication defined via $[ab] = [a] \cdot [b]$ is a commutative group. Moreover, each element $\alpha \neq [1]$ is of order 2, i.e. $\alpha^2 = [1]$ for all $\alpha \in {}^{k^*}/_{(k^*)^2}$. Hence, ${}^{k^*}/_{(k^*)^2}$ has exponent 2. This means in particular, that

- $[a^{-1}] = [a] = [a]^{-1}$,
- $[a]^m = [a]^{m \mod 2}$ and
- $[a]^{i-j} = [a]^{j-i}$ for $m, i, j \in \mathbb{Z}$.

We will write $-\alpha$ instead of $[-1]\alpha$ (notice that $[1] = [-1]$ is possible). Applying these facts to (1), we get the equivalent equation

$$[x^{I-J \mod 2}] = -\alpha_I \alpha_J, \tag{2}$$

where the modulo is applied componentwise. For reading convenience, we will use a minus sign even when calculating modulo 2.

Remark 4.2. For ${}^{k^*}/_{(k^*)^2}$ finite, we know that it is isomorphic to $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_m}$ for some $d_1, ..., d_m \in \mathbb{N}$, since ${}^{k^*}/_{(k^*)^2}$ is abelian. From the fact that ${}^{k^*}/_{(k^*)^2}$ has exponent 2, we can conclude that $d_1 = ... = d_m = 2$. It follows that, if $|{}^{k^*}/_{(k^*)^2}|$ is finite, it is a power of 2 and if $|{}^{k^*}/_{(k^*)^2}| = 2^m$ then ${}^{k^*}/_{(k^*)^2} \cong \mathbb{Z}_2^m$.
In particular, up to isomorphism of groups, it is sufficient to know a field with $|{}^{k^*}/_{(k^*)^2}| = 2^m$ for all $m \in \mathbb{N}$. We will construct such fields in the next Lemma. Note, however, that there still can be important differences between two such quotients, like whether $[1]$ is unequal to $[-1]$ or not.

Lemma 4.3. *Let l be a field and $k = l((t)) = \{\sum_{i=m}^{\infty} c_i t^i \mid c_i \in l,\ m \in \mathbb{Z},\ c_m \neq 0\} \cup \{0\}$ be the field of formal Laurent series over l. Then*

$$|{}^{k^*}/_{(k^*)^2}| = |({}^{l^*}/_{(l^*)^2}) \,\dot\cup\, (t \cdot {}^{l^*}/_{(l^*)^2})| = 2 \cdot |{}^{l^*}/_{(l^*)^2}|.$$

It follows that for ${}^{l^*}/_{(l^*)^2}$ finite $|{}^{k^*}/_{(k^*)^2}| = 2|{}^{l^*}/_{(l^*)^2}|$.
In particular, it is $|\frac{\mathbb{R}((t_1,...,t_m))^*}{(\mathbb{R}((t_1,...,t_m))^*)^2}| = 2^{m+1}$ and for $|{}^{k^*}/_{(k^*)^2}| = 2^{m+1}$ we know from the previous remark that

$${}^{k^*}/_{(k^*)^2} \cong \frac{\mathbb{R}((t_1, ..., t_m))^*}{(\mathbb{R}((t_1, ..., t_m))^*)^2}$$

as groups.

Proof. Recall that in the subring of formal power series $l[[t]] = \{\sum_{i=0}^{\infty} c_i t^i \mid c_i \in l\}$ of $l((t))$ an element is a square if, and only if, c_0 is a square and that the units are exactly the power series with $c_0 \neq 0$. It follows immediately, that in ${l[[t]]^*}/{(l[[t]]^*)^2}$ we have $[\sum_{i=0}^{\infty} c_i t^i] = [c_0 \sum_{i=0}^{\infty} \frac{c_i}{c_0} t^i] = [c_0]$. We conclude that for $[\sum_{i=m}^{\infty} c_i t^i] \in {k^*}/{(k^*)^2}$ there are two cases: Either $m \in 2\mathbb{Z}$ and hence $[\sum_{i=m}^{\infty} c_i t^i] = [t^m(\sum_{i=0}^{\infty} c_{i+m} t^i)] = [\sum_{i=0}^{\infty} c_{i+m} t^i] = [c_m]$ or $m \notin 2\mathbb{Z}$ and hence $[\sum_{i=m}^{\infty} c_i t^i] = [t^m(\sum_{i=0}^{\infty} c_{i+m} t^i)] = [t(\sum_{i=0}^{\infty} c_{i+m} t^i)] = [tc_m]$. □

Example 4.4. We have

$$\left|{\mathbb{R}((t))^*}/{(\mathbb{R}((t))^*)^2}\right| = |\{\pm 1, \pm t\}| = 4$$

and

$$\left|{\mathbb{R}((s,t))^*}/{(\mathbb{R}((s,t))^*)^2}\right| = |\{\pm 1, \pm s, \pm t, \pm st\}| = 8.$$

Remark 4.5. From algebra we know that $|{k^*}/{(k^*)^2}| = 2$ for k a finite field with characteristic unequal to 2. Additionally, it is known from number theory that -1 is a square in a field with characteristic $p = 1 \mod 4$ and not a square in $\mathbb{Z}_p$ with p a prime number and $p = 3 \mod 4$.
If $p = 3 \mod 4$, then in a finite field k of order p^{2n} for some $n \in \mathbb{N}$, -1 is a square: For $GF(p^2) = {\mathbb{Z}_p[x]}/{\langle x^2+1\rangle}$ it is $x^2 = -1$, i.e. -1 is a square. Since 2 is a divisor of $2n$, $GF(p^2)$ is a subfield of $GF(p^{2n})$, hence -1 is a square in this field, too.

Example 4.6. In $\mathbb{Z}_5$ we get $2^2 = -1$, hence -1 is a square. It follows that ${\mathbb{Z}_5^*}/{(\mathbb{Z}_5^*)^2} = \{\{1,4\},\{2,3\}\}$, in particular $[1] = [-1]$.
For $k = \mathbb{Z}_7$ we get that -1 is not a square as $7 = 3 \mod 4$. Hence, ${\mathbb{Z}_7^*}/{(\mathbb{Z}_7^*)^2} = \{[1],[-1]\}$.

Example 4.7. One can calculate that for $k = \mathbb{Z}_3[x]/\langle x^3 + 2x + 1\rangle \cong GF(3^3)$ there is no square root of -1. Hence, ${k^*}/{(k^*)^2} = \{[1],[-1]\}$.

Example 4.8. Obviously, for k algebraically closed, we get ${k^*}/{(k^*)^2} = \{[1]\}$.

Example 4.9. For $k = \mathbb{Q}$ it is

$${\mathbb{Q}^*}/{(\mathbb{Q}^*)^2} = \{[\pm p_1 \cdots p_s] \mid p_i \text{ prime, pairwise different}, s \in \mathbb{N}_0\},$$

in particular $|{\mathbb{Q}^*}/{(\mathbb{Q}^*)^2}| = \infty$.

Example 4.10. In example 2.7. in [PP22] it is shown that

$${k^*}/{(k^*)^2} = {k\{\{t\}\}^*}/{(k\{\{t\}\}^*)^2}.$$

Example 4.11. Consider the hypersurface defined by the over $k = \mathbb{R}((t))$ enriched tropical polynomial $[t]x \oplus [1]y \oplus [-1]0$ in $\mathbb{R}^2$.
For the edge where x and y obtain the maximum, we get from equation (2) that

$$[xy] = [(x,y)^{(1,0)-(0,1) \mod 2}] = -\alpha_{(1,0)}\alpha_{(0,1)} = [-t],$$

respectively

$$[y] = [-t][x].$$

Hence, we get as the set assigned to this edge

$$\{(1,-t),(-1,t),(t,-1),(-t,1)\}.$$

However, since these sets get quite large, we will only note $[t] = \alpha_{(1,0)}\alpha_{(0,1)}$ at the edge (in blue).
Analogously, we get $[-t]$ and $[-1]$ at the other edges, see Figure 3. In particular, we have $[t] \cdot [-1] \cdot [-t] = [1]$.

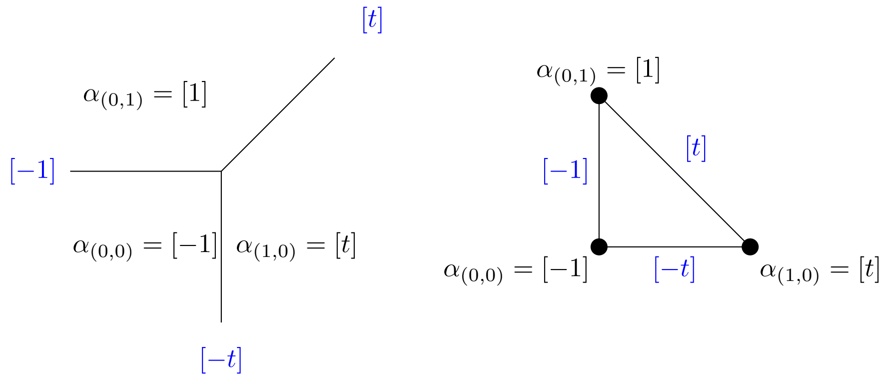


Figure 3: The example for a tropical line in $\mathbb{R}^2$ enriched over $k = \mathbb{R}((t))$ together with its dual from Example 4.11.

Example 4.12. See Figure 4 for two two-dimensional examples of curves with an enrichment α over $k = \mathbb{R}((t))$. At each edge (v, w) we noted the product $\alpha_v\alpha_w$ in blue. Notice that going over any cycle in the dual, the product of the values associated to the edges in the cycle is $[1]$.

Example 4.13. In Figure 5, we see a more complicated example of the dual of a tropical curve this time with an enrichment over $k = \mathbb{R}((s, t))$. Notice again, that going over any cycle, the product of the values associated to the edges in the cycle is $[1]$.

This finally brings us to the definition of a k-phase.

Definition 4.14 (k-phase)**.** A *k-phase* on a non-singular tropical hypersurface C in $\mathbb{R}^n$ is a map $\varphi\colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ such that for each codimension 1 face of C with adjacent facets F_1, F_2, F_3 holds $\varphi(F_1)\varphi(F_2)\varphi(F_3) = [1]$.

For dimensions 2 and 3 this means the following.

Definition 4.15 (k-phase — dimension 2)**.** A *k-phase* on a non-singular tropical hypersurface C in $\mathbb{R}^2$ is a map $\varphi\colon \{\text{edges of } C\} \to {}^{k^*}/_{(k^*)^2}$ such that for each vertex with adjacent edges e_1, e_2, e_3 holds $\varphi(e_1)\varphi(e_2)\varphi(e_3) = [1]$.

Definition 4.16 (k-phase — dimension 3)**.** A *k-phase* on a non-singular tropical hypersurface C in $\mathbb{R}^3$ is a map $\varphi\colon \{\text{two-dimensional cells of } C\} \to {}^{k^*}/_{(k^*)^2}$ such that for each edge with adjacent two-dimensional cells p_1, p_2, p_3 holds $\varphi(p_1)\varphi(p_2)\varphi(p_3) = [1]$.

Remark 4.17. Via duality, we can equivalently interpret a k-phase φ as a map from the set of edges of Δ to ${}^{k^*}/_{(k^*)^2}$. We will write vw (or wv, the order does not matter) for the edge connecting the vertices $v, w \in \Delta_V$.

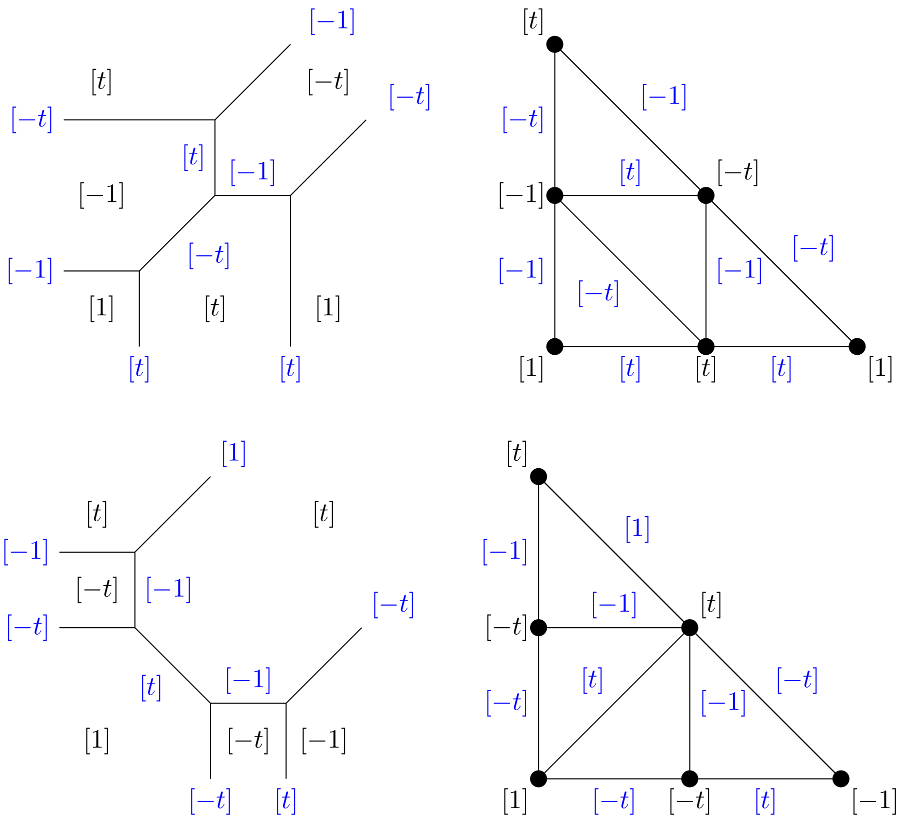


Figure 4: Two tropical hypersurfaces in $\mathbb{R}^2$ enriched over $k = \mathbb{R}((t))$ and their duals, from Example 4.12.

Our first goal is to set enriched hypersurfaces and hypersurfaces with k-phase in connection. To do this, we will proceed analogously to the real phase.

Definition 4.18 (Compatible enrichment)**.** An enrichment α at the vertices of Δ is called *compatible with* φ if for any vertex v of Δ, the following compatibility condition is satisfied:
For any vertex w of Δ adjacent to v, one has $\alpha_v = \varphi(F)\alpha_w$, where $F \in \mathcal{F}(C)$ denotes the facet of C dual to the edge connecting v and w.

The next theorem is the analog to Lemma 3.10. For $k = \mathbb{R}$, we get the statement from Lemma 3.10.

Theorem 4.19. *For any k-phase φ on C, there exist exactly $|{}^{k^*}/_{(k^*)^2}|$ enrichments at the vertices of Δ compatible with φ.*
Reciprocally, given any enrichment α at the vertices of Δ, there exists a unique k-phase φ on C such that φ is compatible with α.

Proof. This proof works just like the one in [Ren17] for lemma 1 (Lemma 3.10 in this text). Let φ be a k-phase on C. Choose an arbitrary vertex v of Δ and put an arbitrary element $\beta \in {}^{k^*}/_{(k^*)^2}$ at v. Given a vertex of Δ equipped with an element of ${}^{k^*}/_{(k^*)^2}$, define an element of ${}^{k^*}/_{(k^*)^2}$ at all adjacent vertices by

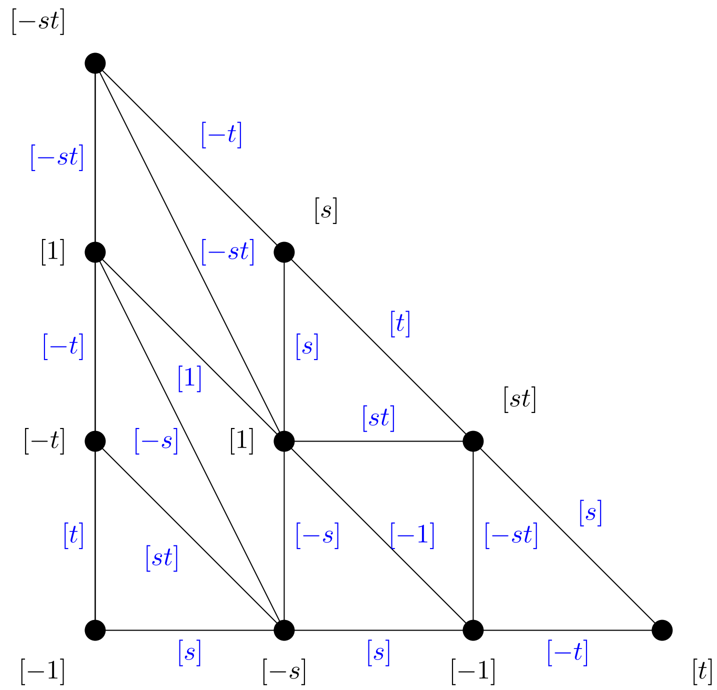


Figure 5: The dual of a more complex tropical curve in $\mathbb{R}^2$ enriched over $k = \mathbb{R}((s,t))$ referenced in Example 4.13.

using the compatibility condition in Definition 4.18. This gives an enrichment α at the vertices of Δ compatible with φ such that $\alpha_v = \beta$. By the definition of k-phase, we know that going over any cycle of length 3 in Δ does not cause a problem. This means exactly that going over any cycle in Δ, we arrive at the same element of ${}^{k^*}/_{(k^*)^2}$, and the enrichment α is well defined.
Reciprocally, let α be an enrichment of Δ. For v, w vertices of Δ, define $\varphi(vw) = \alpha_v\alpha_w$.
We get $\varphi(vw)\varphi(wu)\varphi(uv) = (\alpha_v\alpha_w)(\alpha_w\alpha_u)(\alpha_u\alpha_v) = (\alpha_v)^2(\alpha_w)^2(\alpha_u)^2 = [1]$. □

Now that we have seen that the k-phase behaves like the real phase with respect to an enrichment, our next goal is of course to prove that the two definitions coincide for $k = \mathbb{R}$.
To do this, we will need to translate the k-phase into sets of elements in $({}^{k^*}/_{(k^*)^2})^n$ by looking at the solutions of equations like $[x^I] = \alpha$ in $({}^{k^*}/_{(k^*)^2})^n$. As we will need the theory later anyway, we will actually discuss whole systems of equations.

4.3 Systems of equations in $({}^{k^*}/_{(k^*)^2})^n$

The next few lemmas are quite technical. We are going to set up some theory about systems of equations in $({}^{k^*}/_{(k^*)^2})^n$. It works similarly to systems of linear equations.

Remark 4.20. Let us fix some useful notation. Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in ({}^{k^*}/{}_{(k^*)^2})^n$, $I = (i_1, ..., i_n), v = (v_1, ..., v_n) \in \mathbb{Z}^n$ or $\mathbb{Z}_2^n$ and $\alpha \in {}^{k^*}/{}_{(k^*)^2}$. We define

- $xy = x \cdot y = (x_1y_1, ..., x_ny_n)$,
- $x^I = x_1^{i_1} \cdots x_n^{i_n}$,
- and $\alpha^I = (\alpha^{i_1}, ..., \alpha^{i_n})$.

With this notation we get

- $(\alpha^v)^I = (\alpha^{v_1}, ..., \alpha^{v_n})^I = \alpha^{v_1 i_1} \cdots \alpha^{v_n i_n} = \alpha^{\langle v, I \rangle}$,
- $(x^v)^I = x^{v \cdot I}$ with componentwise multiplication,
- $\alpha^v \cdot \alpha^I = \alpha^{v+I}$, and $x^v \cdot x^I = x^{v+I}$.

For $|{}^{k^*}/{}_{(k^*)^2}| = \infty$, we define $|{}^{k^*}/{}_{(k^*)^2}|^m = \infty$ for any $m \in \mathbb{N}$ and $|{}^{k^*}/{}_{(k^*)^2}|^0 = 1$.
To write down certain subsets of $({}^{k^*}/{}_{(k^*)^2})^n$ more efficiently, we will write $\circ$ for a slot in a tuple that can be filled with any element.
So for example, if ${}^{k^*}/{}_{(k^*)^2} = \{[1], [-1]\}$ we mean by $\{(\circ, [1])\}$ the set

$$\{([1], [1]), ([-1], [1])\}$$

and $\{(\circ, \circ, [1])\}$ is the set of all elements in $({}^{k^*}/{}_{(k^*)^2})^3$ with third entry [1]. By writing something like $\{([\pm 1], [\mp 1])\}$ we mean the set $\{([1], [-1]), ([-1], [1])\}$. Moreover, we will sometimes leave out the brackets around the entries of elements of $({}^{k^*}/{}_{(k^*)^2})^n$ for reading convenience.
All of this will allow us to shorten formulas, computations and formulations vastly.

Next, we give a definition to something that can be understood as the $({}^{k^*}/{}_{(k^*)^2})^n$-analog of a vector subspace in $\mathbb{Z}_2^n$.

Definition 4.21. Let $1 \le m \in \mathbb{N}$, $m \le n \in \mathbb{N}$, $V \subset \mathbb{Z}_2^n$ an m-dimensional vector subspace. We define

$$A_V := \left\{ \begin{pmatrix} \alpha_1^{v_{1,1}} \cdots \alpha_s^{v_{s,1}} \\ \vdots \\ \alpha_1^{v_{1,n}} \cdots \alpha_s^{v_{s,n}} \end{pmatrix} = \alpha_1^{v_1} \cdots \alpha_s^{v_s} \middle| \begin{array}{l} s \in \mathbb{N}, \\ \alpha_i \in {}^{k^*}/{}_{(k^*)^2} \ \forall i = 1, ..., s, \\ v_1, ..., v_s \in V \end{array} \right\}.$$

Lemma 4.22. *For ${}^{k^*}/{}_{(k^*)^2}$ finite holds $|A_V| = |{}^{k^*}/{}_{(k^*)^2}|^m$.*

Proof. Via induction over $\dim V = m$. Let $m = 1$. Then $V = \{0, w\}$ for some $w \in \mathbb{Z}_2^n \backslash \{0\}$. We get for ${}^{k^*}/{}_{(k^*)^2} = \{[1], \alpha_1, ..., \alpha_{p-1}\}$

$$\begin{aligned} A_V &= \{\alpha_1^{v_1} \cdots \alpha_s^{v_s} \,|\, s \in \mathbb{N},\ \alpha_i \in {}^{k^*}/{}_{(k^*)^2} \ \forall i = 1, ..., s;\ v_1, ..., v_s \in V\} \\ &= \left\{\alpha_1^{v_1} \cdots \alpha_{p-1}^{v_{p-1}} \mid v_1, ..., v_{p-1} \in V = \{0, w\}\right\} \\ &= \{\alpha^w \mid \alpha \in {}^{k^*}/{}_{(k^*)^2}\} \end{aligned}$$

and hence $|A_V| = |{}^{k^*}/{}_{(k^*)^2}|$.
Let now $V = \text{span}\{v_1, ..., v_{m+1}\}$ be an $(m + 1)$-dimensional vector subspace

and define $U = \mathrm{span}\{v_1, ..., v_m\}$ and $W = \mathrm{span}\{v_{m+1}\}$. Then $\dim U = m$ and $\dim W = 1$. Since $V = U \oplus W$ is a direct sum, the map $\psi : A_U \times A_W \to A_V$ defined via

$$\left(\alpha_1^{u_1} \cdots \alpha_{p-1}^{u_{p-1}},\ \alpha_1^{w_1} \cdots \alpha_{p-1}^{w_{p-1}}\right) \mapsto \alpha_1^{u_1+w_1} \cdots \alpha_{p-1}^{u_{p-1}+w_{p-1}}$$

is a well-defined bijection. By the induction assumption, we get $|A_V| = |A_U \times A_W| = |A_U| \cdot |A_W| = |{}^{k^*}/_{(k^*)^2}|^m \cdot |{}^{k^*}/_{(k^*)^2}| = |{}^{k^*}/_{(k^*)^2}|^{m+1}$. □

Remark 4.23. For $k = \mathbb{R}$ and $y \in ({}^{k^*}/_{(k^*)^2})^n$, the set $yA_{\mathrm{span}\{I-J\}^\perp}$ is exactly the set we get if we alter y in all possible ways given by the first property in Definition 3.3 of real phase (with $I - J$ the corresponding vector to the facet). By Lemma 4.22, we get that this is actually the whole assigned set.

Lemma 4.24. *Let $w_1, ..., w_m \in \mathbb{Z}_2^n$, $V := \mathrm{span}\{w_1, ..., w_m\}$, $\alpha_1, ..., \alpha_m \in {}^{k^*}/_{(k^*)^2}$. If $y \in ({}^{k^*}/_{(k^*)^2})^n$ solves*

$$\begin{cases} \alpha_1 = x^{w_1} \\ \quad\vdots \\ \alpha_m = x^{w_m} \end{cases},$$

then so does every element of $yA_{V^\perp}$.

Proof. For $j \in \{1, ..., m\}$, $\beta_1, ..., \beta_s \in {}^{k^*}/_{(k^*)^2}$, $v_1, ..., v_s \in V^\perp$ we get:

$$\begin{aligned}(\beta_1^{v_1} \cdots \beta_s^{v_s} y)^{w_j} &= (\beta_1^{v_1} \cdots \beta_s^{v_s})^{w_j} y^{w_j} \\ &= \beta_1^{\langle v_1, w_j\rangle} \cdots \beta_s^{\langle v_s, w_j\rangle} y^{w_j} = \beta_1^0 \cdots \beta_s^0 y^{w_j} = y^{w_j} = \alpha_j.\end{aligned}$$

□

Lemma 4.25. *Let $w_1, ..., w_m \in \mathbb{Z}_2^n$, $V := \mathrm{span}\{w_1, ..., w_m\}$, $p = \dim V$, $\alpha_1, ..., \alpha_m \in {}^{k^*}/_{(k^*)^2}$. Moreover, let $w_1, ..., w_p$ be a basis of V and $w_l = \lambda_{l,1} w_1 + ... + \lambda_{l,p} w_p$. The system of equations*

$$\begin{cases} \alpha_1 = x^{w_1} \\ \quad\vdots \\ \alpha_m = x^{w_m} \end{cases}, \tag{3}$$

has a solution if, and only if, $\alpha_l = \alpha_1^{\lambda_{l,1}} \cdots \alpha_p^{\lambda_{l,p}}$ for all $l = p + 1, ..., m$.
If y is a solution, then the set of solutions is $yA_{V^\perp}$. In particular, there are either $|{}^{k^*}/_{(k^*)^2}|^{n-p}$ solutions or 0.

Proof. The first statement follows from the equation

$$\alpha_l = x^{w_l} = x^{\lambda_{l,1} w_1 + ... + \lambda_{l,p} w_p} = x^{\lambda_{l,1} w_1} \cdots x^{\lambda_{l,p} w_p} = \alpha_1^{\lambda_{l,1}} \cdots \alpha_p^{\lambda_{l,p}}.$$

By transforming the equations like $\alpha_j = x^{w_j} \Leftrightarrow \alpha_j \alpha_l = x^{w_j + w_l}$, we can w.l.o.g. assume that the matrix with rows $w_1, ..., w_p$ is in reduced row form. From this we see that there are exactly $n - p$ free parameters and we get that there exist $|{}^{k^*}/_{(k^*)^2}|^{n-p}$ solutions to the first p equations if ${}^{k^*}/_{(k^*)^2}$ is finite. Hence, if there is a solution to all equations, there are $|{}^{k^*}/_{(k^*)^2}|^{n-p}$ for ${}^{k^*}/_{(k^*)^2}$ finite.

By Lemma 4.24 follows that the set of solutions of (3) is a super set of $yA_{V^\perp}$. For ${}^{k^*}/{(k^*)^2}$ finite, equality follows from the equality of the cardinalities (Lemma 4.22).
Let now $(z_1, ..., z_n)$ be a solution to (3). In particular, it solves this system of equations too, if we consider it in the finite subgroup G of ${}^{k^*}/{(k^*)^2}$ generated by $\{z_1, ..., z_n, \alpha_1, ..., \alpha_m\}$. As we have already proven the finite case, it follows that z is an element of $A_{V^\perp} \cap G \subseteq A_{V^\perp}$. $\square$

Remark 4.26. Consider the map

$$\eta\colon\ (({}^{k^*}/{(k^*)^2})^n, \cdot) \to (({}^{k^*}/{(k^*)^2})^m, \cdot)\,; x \mapsto (x^{w_1}, ..., x^{w_m})$$

with $w_1, ..., w_m \in \mathbb{Z}_2^n$ as before. This is obviously a group homomorphism. What we have proven in Lemma 4.25 is that $\ker \eta = A_{\operatorname{span}\{w_1,...,w_m\}^\perp}$ and that the sets of solutions of such systems of equations are cosets of $\ker \eta$ in $({}^{k^*}/{(k^*)^2})^n$. More precisely, we have that

$$\{\text{solutions to (3)}\} = \eta^{-1}(\alpha_1, ..., \alpha_m) = y \ker \eta \text{ for any } y \in \eta^{-1}(\alpha_1, ..., \alpha_m).$$

Hence, similarly to systems of linear equations, we see that the solutions of such systems of equations form some kind of "affine subspace" in $({}^{k^*}/{(k^*)^2})^n$.

The next three lemmas are all direct consequences of Lemma 4.25. Outsourcing them is mainly for convenience in some following proofs.

Lemma 4.27. *Let $I \in \mathbb{Z}_2^n,\ \alpha \in {}^{k^*}/{(k^*)^2}$. An equation of the form*

$$\alpha = x^I = x_1^{i_1} \cdot ... \cdot x_n^{i_n}$$

*in ${}^{k^*}/{(k^*)^2}$ has $|{}^{k^*}/{(k^*)^2}|^{n-1}$ solutions if $I \neq 0$. If $I = 0$ it has 0 solutions for $\alpha \neq [1]$ and $|{}^{k^*}/{(k^*)^2}|^n$ for $\alpha = [1]$.
If $i_j = 1$, then $y \in ({}^{k^*}/{(k^*)^2})^n$ defined via $y_j = \alpha$, $y_l = [1]$ for $l \neq j$ is a solution and the set of solutions equals $yA_{I^\perp}$ for $I^\perp := \{v \in \mathbb{Z}_2^n \mid \langle v, I\rangle = 0\}$.*

The following lemma will be important when we talk about intersections, too. It solves the problem of a system of two equations.

Lemma 4.28. *Let $\alpha, \beta \in {}^{k^*}/{(k^*)^2},\ n \geq 2,\ I, J \in \mathbb{Z}_2^n$. Consider the system of equations*

$$\begin{cases} \alpha = x^I = x_1^{i_1} \cdot ... \cdot x_n^{i_n} \\ \beta = x^J = x_1^{j_1} \cdot ... \cdot x_n^{j_n} \end{cases}. \tag{4}$$

The set of solutions has

- $|{}^{k^*}/{(k^*)^2}|^{n-2}$ *elements if, and only if, $I \neq J$ with $I, J \neq 0$.*
- $|{}^{k^*}/{(k^*)^2}|^{n-1}$ *elements if, and only if,*
  - *i) $I = J \neq 0$ and $\alpha = \beta$ or*
  - *ii) $0 = I \neq J$ and $\alpha = [1]$ or $0 = J \neq I$ and $\beta = [1]$*
- 0 *elements if, and only if,*
  - *i) $I = J \neq 0$ and $\alpha \neq \beta$ or*

ii) $I = J = 0$ *and* $\alpha \neq [1]$ *or* $\beta \neq [1]$ *or*

iii) $0 = I \neq J$ *and* $\alpha \neq [1]$ *or* $0 = J \neq I$ *and* $\beta \neq [1]$

- $|{}^{k^*}/{}_{(k^*)^2}|^n$ *elements if, and only if,* $I = J = 0$ *and* $\alpha = \beta = [1]$.

The next lemma deals with systems of three equations.

Lemma 4.29. *Let* $I, J, L \in \mathbb{Z}_2^n \backslash \{0\}$ *be pairwise different such that* $I+J+L = 0$, $\alpha, \beta, \gamma \in {}^{k^*}/{}_{(k^*)^2}$. *Then the system of equations*

$$\begin{cases} -\alpha = x^I \\ -\beta = x^J \\ -\gamma = x^L \end{cases} \tag{5}$$

has no solution if, and only if, $\alpha\beta\gamma \neq [-1]$.
In particular, it has no solution for $\alpha\beta\gamma = [1]$ *if* $[1] \neq [-1]$ *in* ${}^{k^*}/{}_{(k^*)^2}$.

4.4 Equivalence of $\mathbb{R}$-phase and real phase

We want to recall what we did in Example 4.11 and get sets of elements in $({}^{k^*}/{}_{(k^*)^2})^n$ from a k-phase. Let φ be a k-phase on C, $F \in \mathcal{F}(C)$ with corresponding vector $I - J \in \mathbb{Z}^n$ and $\varphi(F) = \alpha$. Then, we get the equation $[x^I] = [-a][x^J]$ in ${}^{k^*}/{}_{(k^*)^2}$, or equivalently

$$-\alpha = [x^{I-J}] = [x^{(I-J) \mod 2}].$$

By Lemma 4.27, this equation has always exactly $|{}^{k^*}/{}_{(k^*)^2}|^{n-1}$ or 0 solutions. For C non-singular, the case of 0 solutions is not possible, as the next lemma implies.

Lemma 4.30. *Let* C *be a non-singular tropical hypersurface in* $\mathbb{R}^n$ *and* $F \in \mathcal{F}(C)$. *Then for the corresponding vector* $I - J$ *it is* $I - J \mod 2 \neq 0$.

Proof. As the dual subdivision of a non-singular tropical hypersurface only consists of unimodular simplices, they also only have unimodular faces, in particular unimodular edges. Assume $I - J \mod 2 = 0$. Then the edge in the dual corresponding to F is not unimodular, a contradiction. □

In conclusion, we can interpret a k-phase $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/{}_{(k^*)^2}$ as a map into the subsets of $({}^{k^*}/{}_{(k^*)^2})^n$ with $|{}^{k^*}/{}_{(k^*)^2}|^{n-1}$ elements. We will write φ_F for the set corresponding to $\varphi(F)$.

Remark 4.31. Recall Definition 4.18 of compatible enrichments and Definition 3.9 of compatible distributions of signs. As $(+, ..., +) \hat{=} ([1], ..., [1])$ is a solution of $x^{I-J} = -\varphi(F)$, $I - J \neq 0 \mod 2$, if, and only if, $\varphi(F) = [-1]$, we get that they coincide for $k = \mathbb{R}$.

Now we can prove that $\mathbb{R}$-phase and real phase are equivalent.

Theorem 4.32. *Let* C *be a non-singular tropical hypersurface in* $\mathbb{R}^n$. *To each real phase* ϕ *on* C *there exists a unique corresponding* $\mathbb{R}$*-phase* φ *and to each* $\mathbb{R}$*-phase* φ *on* C *there exists a unique corresponding real phase* ϕ.

Consequentially, we will use real phase and $\mathbb{R}$-phase as synonyms from now on.

Proof. Let C have a $\mathbb{R}$-phase $\varphi\colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$. Let $F \in \mathcal{F}(C)$ with corresponding vector $I - J$. Define $\phi_F = \{\phi_{F,i} = \left(\phi^1_{F,i}, ..., \phi^n_{F,i}\right) \mid 1 \leq i \leq 2^{n-1}\}$ to be the solutions to the equation

$$-\varphi(F) = x^{I-J}. \tag{6}$$

This is well-defined by Lemma 4.27.
First property: Let v be an integer vector in the direction of F. Then v is orthogonal to $I - J$, i.e. $v_1(i_1 - j_1) + ... + v_n(i_n - j_n) = 0$. If now $\phi_{F,i} = \left(\phi^1_{F,i}, ..., \phi^n_{F,i}\right)$ solves (6), we get for $\phi_{F,j}$ defined via $\phi^k_{F,j} = [-1]^{v_k}\varphi^k_{F,i}$, $1 \leq k \leq n$ that

$$\begin{aligned}
(\phi_{F,j})^{I-J} &= (\phi^1_{F,j})^{i_1-j_1} \cdots (\phi^n_{F,j})^{i_n-j_n} \\
&= [-1]^{v_1(i_1-j_1)}(\phi^1_{F,i})^{i_1-j_1} \cdots [-1]^{v_n(i_n-j_n)}(\phi^n_{F,i})^{i_n-j_n} \\
&= [-1]^{v_1(i_1-j_1)+...+v_n(i_n-j_n)}(\phi_{F,i})^{I-J} \\
&= [-1]^0(\phi_{F,i})^{I-J} = -\varphi(F)
\end{aligned}$$

as desired.
Second Property: Let H be a codimension 1 face of C with adjacent facets F_1, F_2, F_3 corresponding to $I - J, J - L, L - I \in \mathbb{Z}^n$ and $\alpha_j = \varphi(F_j)$. By Lemma 4.29, we know that there is no n-tuple that is an element of ϕ_{F_1}, ϕ_{F_2} and ϕ_{F_3}. Hence, we get the uniqueness. It remains to show: If y solves $-\alpha_1 = x^{I-J}$ but not $-\alpha_3 = x^{L-I}$, then y solves $-\alpha_2 = x^{J-L}$. As $-\alpha_3 \neq y^{L-I}$ and $|{}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2}| = |\{[1], [-1]\}| = 2$, we have $\alpha_3 = y^{L-I}$. Hence, we get $y^{J-L} = y^{L-J} = y^{L-I} \cdot y^{I-J} = \alpha_3 \cdot (-\alpha_1) = -\alpha_2$, since $\alpha_1\alpha_2\alpha_3 = [1]$.

Let C have a real phase ϕ. Define $\varphi(F) = [-1]$ if $([1], ..., [1]) \in \phi_F$ and $\varphi(F) = [1]$ if $([1], ..., [1]) \notin \phi_F$. Let H be a codimension 1 face of C with adjacent facets F_1, F_2, F_3. By the second property, we know that $([1], ..., [1])$ is either an element of two or none of $\phi_{F_1}, \phi_{F_2}, \phi_{F_3}$. Hence, we get $\varphi_{F,1}\varphi_{F,2}\varphi_{F,3} = [1]$ and are done. □

4.5 Alternative Definition for k-phase

In this subsection, we will give an alternative definition of a k-phase on a non-singular tropical hypersurface that is more analogous to the real phase definition. However, as we will see later, the firstly given notation and definition for k-phase will be easier to generalize further.
The reason why we will give this alternative definition anyway, is that it helps to understand the connection of k-phase and real phase better, in particular how the two properties from the real phase definition come out of the one from the k-phase. Since this is our only goal in this subsection, we will restrict, for this subsection only, to fields for which ${}^{k^*}/_{(k^*)^2}$ is finite.

Definition 4.33 (k-phase)**.** Let $p = |{}^{k^*}/_{(k^*)^2}| < \infty$. A *$k$-phase* on a non-singular tropical hypersurface in $\mathbb{R}^n$ is the data for every facet $F \in \mathcal{F}(C)$ of p^{n-1} n-tuples of ${}^{k^*}/_{(k^*)^2}$-elements $\varphi_{F,i} = \left(\varphi^1_{F,i}, ..., \varphi^n_{F,i}\right)$, $1 \leq i \leq |{}^{k^*}/_{(k^*)^2}|^{n-1}$ satisfying the following properties:

1. If $1 \leq i \leq p^{n-1}$ and $v = (v_1, ..., v_n)$ is an integer vector in the direction of F, then there exists $1 \leq j \leq p^{n-1}$ such that $\alpha^{v_k} \varphi^k_{F,i} = \varphi^k_{F,j}$, for $1 \leq k \leq n$ and $\alpha \in {}^{k^*}/_{(k^*)^2}$.

2. Let H be a codimension 1 face of C with adjacent facets F_1, F_2, F_3 corresponding to $I-J, J-L, L-I \in \mathbb{Z}_2^n$. Then $(\varphi_{F_1,1})^{I-J}(\varphi_{F_2,1})^{J-L}(\varphi_{F_3,1})^{L-I} = [-1]$.

   Define $\varphi_F = \{\varphi_{F,i} = \left(\varphi^1_{F,i}, ..., \varphi^n_{F,i}\right) \mid 1 \leq i \leq |{}^{k^*}/_{(k^*)^2}|^{n-1}\}$.

Remark 4.34. From Lemma 4.22 we deduce that for a k-phase as defined in 4.33 the first property and one element of $({}^{k^*}/_{(k^*)^2})^n$ completely define the set φ_F.

Proof. Let $H \subseteq \mathbb{Z}_2^n$ be the by Lemma 4.30 $(n-1)$-dimensional vector subspace defined by taking the componentwise modulo 2 of the integer vectors in the directions of F and $\varphi_{F,1} \in \varphi_F$. By property 1 $A_H \cdot \varphi_{F,1} \subseteq \varphi_F$. Equality follows by comparing the cardinality of the sets. □

Example 4.35. In Figure 6 we see a three-dimensional example of a tropical hypersurface with enrichment α over $k = \mathbb{R}((s,t))$ and both versions of the same k-phase φ.

Remark 4.36. This definition is written as much in the style of Definition 3.3 of real phase as possible. With the notation we have established, we can equivalently write:
A *k-phase structure* on a non-singular tropical hypersurface C in $\mathbb{R}^n$ is the data for every facet $F \in \mathcal{F}$ with corresponding vector $I - J$ of a set $y_F A_{(I-J)^\perp} \subseteq ({}^{k^*}/_{(k^*)^2})^n$ such that for H a codimension 1 face of C with adjacent facets F_1, F_2, F_3 corresponding to $I-J, J-L, L-I \in \mathbb{Z}_2^n$ holds $y_{F_1}^{I-J} y_{F_2}^{J-L} y_{F_3}^{L-I} = [-1]$. For this definition, ${}^{k^*}/_{(k^*)^2}$ finite is not needed.

The following lemma is needed to prove that for three adjacent facets the associated sets have empty intersection.

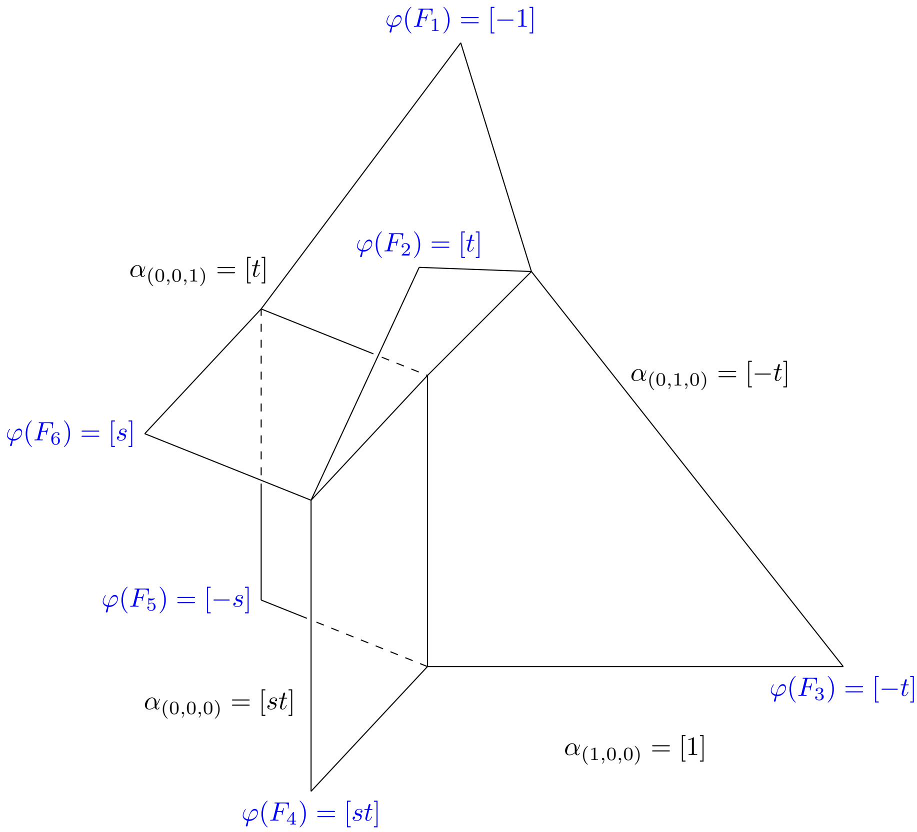
Lemma 4.37. *Let C be a non-singular tropical hypersurface in $\mathbb{R}^n$ and H be a codimension 1 face of C with adjacent facets F_1, F_2, F_3 corresponding to $I-J, J-L, L-I \in \mathbb{Z}^n$. Then $(I-J) \mod 2, (J-L) \mod 2$ and $(L-I) \mod 2$ are pairwise different.*

Proof. Equip C with the real phase $\varphi(F) = [1]$ for all facets F. If w.l.o.g. $(I-J) \mod 2 = (J-L) \mod 2$, the defining equations $[-1] = x^{I-J}$ and $[-1] = x^{J-L}$ would coincide, a contradiction. □

Next, we want to outsource a simple observation about the second property in the definition of real phase. We state two more equivalent versions of it.

Lemma 4.38. *Let A_1, A_2, A_3 be finite sets of the same cardinality m. Then the following three properties are equivalent:*

- $\forall i \in \{1,2,3\} \colon \forall a \in A_i \ \exists! j \in \{1,2,3\} \backslash \{i\} \colon a \in A_j$
- $A_1 = (A_1 \cap A_2) \,\dot\cup\, (A_1 \cap A_3)$*,* $A_2 = (A_1 \cap A_2) \,\dot\cup\, (A_2 \cap A_3)$ *and* $A_3 = (A_1 \cap A_3) \,\dot\cup\, (A_2 \cap A_3)$



- $\varphi_{F_1} = \{(\circ, \pm 1, \pm 1), (\circ, \pm t, \pm t), (\circ, \pm s, \pm s), (\circ, \pm st, \pm st)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [1] = yz\}$
- $\varphi_{F_2} = \{(\pm 1, \circ, \mp t), (\pm t, \circ, \mp 1), (\pm s, \circ, \mp st), (\pm st, \circ, \mp s)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [-t] = xz\}$
- $\varphi_{F_3} = \{(\pm 1, \pm t, \circ), (\pm t, \pm 1, \circ), (\pm s, \pm st, \circ), (\pm st, \pm s, \circ)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [t] = xy\}$
- $\varphi_{F_4} = \{(-st, \circ, \circ)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [-st] = x\}$
- $\varphi_{F_5} = \{(\circ, s, \circ)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [s] = y\}$
- $\varphi_{F_6} = \{(\circ, \circ, -s)\}$
  $= \{(x, y, z) \in (\mathbb{R}((st))^* / (\mathbb{R}((st))^*)^2)^3 \mid [-s] = z\}$

Figure 6: The over $k = \mathbb{R}((s, t))$ enriched tropical hyperplane in $\mathbb{R}^3$ with compatible k-phase from Example 4.35.

- $A_1 \cap A_2 \cap A_3 = \emptyset$ *and* $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = \frac{m}{2} \in \mathbb{N}$.

Proof. The equivalence of the first two statements is obvious.
Second to third: $A_1 \cap A_2 \cap A_3 = \emptyset$ must be true because the unions are disjoint. The three equations of sets give a system of linear equations for $|A_i \cap A_j|, i \neq j$ with unique solution $|A_i \cap A_j| = \frac{m}{2}, \ \forall i \neq j$.
Third to second: Obviously, $A_1 \supset (A_1 \cap A_2) \dot{\cup} (A_1 \cap A_3)$, where the union is disjoint because $A_1 \cap A_2 \cap A_3 = \emptyset$. Equality follows from $|A_1 \cap A_2| = |A_1 \cap A_3| = \frac{m}{2}$. Analogously for A_2, A_3. □

With this preparation, we can outline how one gets from Definition 4.33 to Definition 3.3 of real phase.

Remark 4.39. Let φ be a k-phase on a non-singular tropical hypersurface. By Lemma 4.37 we know that all occurring exponents $I - J \mod 2, J - L \mod 2, L - I \mod 2$ for $I - J, J - L, L - I$ corresponding vectors to three facets F_1, F_2, F_3 adjacent to the same codimension 1 face, are pairwise different.
Applying Lemma 4.29, we get that $\varphi_{F_1} \cap \varphi_{F_2} \cap \varphi_{F_3} = \emptyset$ if, and only if, $[-1] \neq [1]$ in ${k^*}/{(k^*)^2}$.
Lemma 4.28 tells us that $|\varphi_{F_i} \cap \varphi_{F_j}| = |{k^*}/{(k^*)^2}|^{n-2}$ for all $i \neq j$. Hence, via Lemma 4.38, we get that this implies property 2 from Definition 3.3 of real phase only for $|{k^*}/{(k^*)^2}| = |\{[1], [-1]\}| = 2$.
If $|{k^*}/{(k^*)^2}| > 2$ or $[1] = [-1]$, we get that $\varphi_{F_1} \supsetneq (\varphi_{F_1} \cap \varphi_{F_2}) \cup (\varphi_{F_1} \cap \varphi_{F_3})$, as the cardinalities do not add up.
To sum it up: The first property of the k-phase as defined in 4.33 comes from the way the equations in ${k^*}/{(k^*)^2}$ and ${k^*}/{(k^*)^2}$ itself are structured. The cardinality of the sets comes from that and the fact that the tropical hypersurface is considered to be non-singular.
The uniqueness in the second property of real phase comes from all of that and the fact that $[1] \neq [-1]$ in ${\mathbb{R}^*}/{(\mathbb{R}^*)^2}$, the existence stems from all of that and $|{\mathbb{R}^*}/{(\mathbb{R}^*)^2}| = 2$.

Example 4.40. Let $k = \mathbb{Z}_5$. Then ${k^*}/{(k^*)^2}$ has two elements, namely $[1] = \{1, 4\}$ and $[2] = \{2, 3\}$, in particular $[1] = [-1]$. Consider the tropical line in $\mathbb{R}^2$ with k-phase assigning to each facet the element $[1]$ (Figure 7). Then

- $\varphi_{F_1} = \{(x, y) \in ({k^*}/{(k^*)^2})^2 \mid y = [1]\} = \{([1], [1]), ([2], [1])\}$,
- $\varphi_{F_2} = \{(x, y) \in ({k^*}/{(k^*)^2})^2 \mid xy = [1]\} = \{([1], [1]), ([2], [2])\}$, and
- $\varphi_{F_3} = \{(x, y) \in ({k^*}/{(k^*)^2})^2 \mid x = [1]\} = \{([1], [1]), ([1], [2])\}$.

This demonstrates that for $[1] = [-1]$ in ${k^*}/{(k^*)^2}$ it is $\varphi_{F_1} \cap \varphi_{F_2} \cap \varphi_{F_3} \neq \emptyset$. Moreover, we have

$$\varphi_{F_1} = \{([1], [1]), ([2], [1])\} \supsetneq \{([1], [1])\} = (\varphi_{F_1} \cap \varphi_{F_2}) \cup (\varphi_{F_1} \cap \varphi_{F_3}).$$

Example 4.41. For the curve with $\mathbb{R}((t))$-phase we see in Figure 8 it is

$$\varphi_{F_1} = \{(\circ, [-t])\} \supsetneq \{([t], [-t])\} \dot{\cup} \{([-t], [-t])\} = (\varphi_{F_1} \cap \varphi_{F_2}) \dot{\cup} (\varphi_{F_1} \cap \varphi_{F_3}),$$

as $|{\mathbb{R}((t))^*}/{(\mathbb{R}((t))^*)^2}| = 4 > 2$.

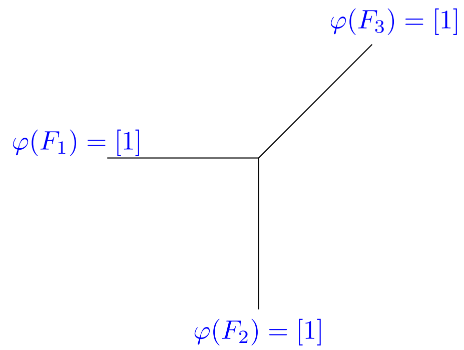


Figure 7: Example 4.40's tropical line in $\mathbb{R}^2$ enriched over $k = \mathbb{Z}_5$ such that $\varphi_{F_1} \cap \varphi_{F_2} \cap \varphi_{F_3} \neq \emptyset$.

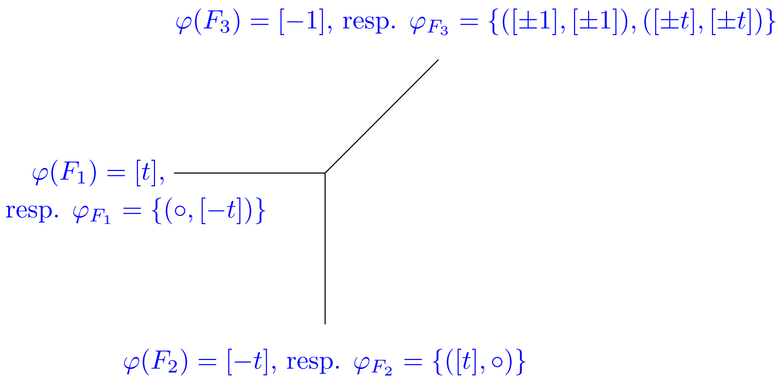


Figure 8: The in Example 4.41 referenced example for a tropical line in $\mathbb{R}^2$ with a k-phase for $k = \mathbb{R}((t))$.

Now that we have given two different definitions of k-phase, we of course have to prove that they are equivalent.

Theorem 4.42. *Let C be a non-singular tropical hypersurface in $\mathbb{R}^n$. Let $F \in \mathcal{F}(C)$ with corresponding vector $I - J$ and $|{}^{k^*}/_{(k^*)^2}| = p$.*

If $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ is a k-phase as defined in 4.14, then the sets $\varphi_F := \{x \in ({}^{k^*}/_{(k^*)^2})^n \mid -\varphi(F) = x^{I-J},\ I - J$ the corresponding vector to $F\}$ satisfy Definition 4.33.

If $\{\varphi_F = \{\varphi_{F,1}, ..., \varphi_{F,p^{n-1}}\} \mid F \in \mathcal{F}(C)\}$ is a k-phase as defined in 4.33, then $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}; F \mapsto -(\varphi_{F,1})^{I-J}$ is well defined and a k-phase as defined in 4.14, $I - J$ again is the corresponding vector to F.

Proof. Let $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ be a k-phase as defined in 4.14.
By Lemma 4.27 it is $|\varphi_F| = |{}^{k^*}/_{(k^*)^2}|^{n-1} = p^{n-1}$.
Proof of first property: Let v be an integer vector in the direction of the facet F and $I - J$ be the facet's corresponding vector. Then v is orthogonal to $I - J$, i.e. $\langle I - J, v\rangle = 0$. If now $\left(\varphi^1_{F,i}, ..., \varphi^n_{F,i}\right)$ solves $-\varphi(F) = x^{I-J}$, we have for $\varphi_{F,j}$

defined via $\varphi_{F,j} = \alpha^v \varphi_{F,i}$ that

$$(\varphi_{F,j})^{I-J} = (\alpha^v \varphi_{F,i})^{I-J} = (\alpha^v)^{I-J} (\varphi_{F,i})^{I-J}$$
$$= \alpha^{\langle v, I-J \rangle} (\varphi_{F,i})^{I-J} = (\varphi_{F,i})^{I-J} = -\varphi(F).$$

as desired.
Proof of second property: Let F_1, F_2, F_3 be three different facets of C adjacent to the same codimension 1 face, corresponding to the exponents $I-J, J-L, L-I$. We compute $(\varphi_{F_1,1})^{I-J}(\varphi_{F_2,1})^{J-L}(\varphi_{F_3,1})^{L-I} = (-\varphi(F_1))(-\varphi(F_2))(-\varphi(F_3)) = [-1]$.

Let $\{\varphi_F \mid F \text{ facet of } C\}$ be a k-phase as defined in 4.33 and ${}^{k^*}/_{(k^*)^2} = \{[1], \alpha_1, ..., \alpha_{p-1}\}$. Let $F \in \mathcal{F}(C)$ be a facet of C and $I-J$ be the corresponding vector. Define $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}; F \mapsto -(\varphi_{F,1})^{I-J}$. This is well defined, i.e. does not depend on the choice of $\varphi_{F,1}$ as by Remark 4.34 any other $\varphi_{F,j} \in \varphi_F$ is of the form

$$\varphi_{F,j} = \alpha_1^{v_1} \cdots \alpha_{p-1}^{v_{p-1}} \varphi_{F,1}$$

for some integer vectors $v_1, ..., v_{p-1}$ in the direction of F and hence

$$(\varphi_{F,j})^{I-J} = \left(\alpha_1^{v_1} \cdots \alpha_{p-1}^{v_{p-1}} \varphi_{F,1}\right)^{I-J} = (\alpha_1^{v_1})^{I-J} \cdots (\alpha_{p-1}^{v_{p-1}})^{I-J} \varphi_{F,1}^{I-J}$$
$$= (\alpha_1^{v_1})^{I-J} \cdots (\alpha_{p-1}^{v_{p-1}})^{I-J} \varphi_{F,1}^{I-J} = \alpha_1^{\langle v_1, I-J \rangle} \cdots \alpha_{p-1}^{\langle v_{p-1}, I-J \rangle} \varphi_{F,1}^{I-J}$$
$$= \alpha_1^0 \cdots \alpha_{p-1}^0 \varphi_{F,1}^{I-J}$$
$$= \varphi_{F,1}^{I-J}.$$

Finally, we see that

$$\varphi(F_1)\varphi(F_2)\varphi(F_3) = (-(\varphi_{F_1,1})^{I-J})(-(\varphi_{F_2,1})^{J-L})(-(\varphi_{F_3,1})^{L-I}) = [1].$$

□

4.6 k-phase for singular hypersurfaces

Recall Definition 4.14:

Definition. A *k-phase* on a non-singular tropical hypersurface C in $\mathbb{R}^n$ is a map $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ such that for each codimension 1 face of C with adjacent facets F_1, F_2, F_3 holds $\varphi(F_1)\varphi(F_2)\varphi(F_3) = [1]$.

If we look closely, we notice that for this definition we only needed the property of C to be non-singular to have exactly three adjacent facets to a codimension 1 facet. If we alter the definition a little bit, we don't actually need this property.

Definition 4.43 (k-phase)**.** A *k-phase* on a tropical hypersurface C in $\mathbb{R}^n$ is a map $\varphi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ such that for each codimension 1 face of C with adjacent facets $F_1, ..., F_p$ holds $\varphi(F_1) \cdots \varphi(F_p) = [1]$.

Remark 4.44. It might be worth noticing that, given a tropical hypersurface C in $\mathbb{R}^n$, the set of k-phase structures on C is a group via point-wise multiplication. That is, if we have two k-phases $\varphi, \psi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ we can define their product $\varphi \cdot \psi \colon \mathcal{F}(C) \to {}^{k^*}/_{(k^*)^2}$ via $\varphi \cdot \psi(F) := \varphi(F)\psi(F)$. As one can easily check, this is again a k-phase.
This group is obviously abelian and has exponent 2.

Obviously, this definition coincides with the one for non-singular hypersurfaces for such. With the same procedure as in Theorem 4.19, we get such a k-phase φ from an enrichment α: Define $\varphi(vw) = \alpha_v \alpha_w$ for v, w vertices of the dual subdivision Δ.
Then, for $F_1, ..., F_p$ adjacent to the same codimension 1 face, every α_v occurs twice in the product $\varphi(F_1) \cdots \varphi(F_p)$ and hence the product is [1].
However, the sets of solutions for the equations $[x^{I-J}] = -\varphi_F$ may behave rather strange, as, for example, they can happen to be empty: If for a $F \in \mathcal{F}(C)$ the corresponding vector is $I - J = 0 \mod 2$ and $\varphi(F) \neq [1]$. Or, in the other extreme, if $I - J = 0 \mod 2$ and $\varphi(F) = [1]$, the set is whole $({}^{k^*}/{}_{(k^*)^2})^n$.
Here we notice an advantage of our way to define a k-phase not as sets, if the set is not of the wanted form, we do not run into a problem.
Of course, the same question arises as when we firstly defined the k-phase for non-singular hypersurfaces: Do we get a $|{}^{k^*}/{}_{(k^*)^2}|$ to 1 correspondence for this case, too? We will proceed with proving that the answer to this question is yes.
To do so, we need the following fact and notation: For a subdivision Δ dual to a tropical hypersurface we denote by Δ_V the set of vertices and by Δ_E the set of edges. For such a Δ there always exists a non-singular subdivision $\Delta' = (\Delta'_V, \Delta'_E)$ such that $\Delta_V \subseteq \Delta'_V$ and $\Delta_E \subseteq \Delta'_E$.

Lemma 4.45. *Let φ be a k-phase on a subdivision Δ dual to a tropical hypersurface. Let $\Delta' = (\Delta'_V, \Delta'_E)$ be a non-singular subdivision such that $\Delta_V \subseteq \Delta'_V$ and $\Delta_E \subseteq \Delta'_E$.*
Then there exists a k-phase φ' on Δ' such that $\varphi'(e) = \varphi(e)$ for all edges e of Δ.

See Figure 9 for an example of how this proof works. There, the edges and the vertex of Δ' which do not have an associated element of ${}^{k^*}/{}_{(k^*)^2}$ yet, are marked red.

Proof. For C a cycle using the edges $e_1, ..., e_n$ we call $\prod_{l=1}^{n} \varphi(e_l)$ the product value of C.
If $\Delta' = \Delta$ there is nothing to prove. So let $\Delta' \neq \Delta$. Let $i = |\Delta'_V \setminus \Delta_V|$ be the number of vertices and $j = |\Delta'_E \setminus \Delta_E|$ be the number of edges added to Δ.
We use induction over i. For the base case, let $i = 0$. We use induction over j. So let now $j = 1$. In this case, there exist vertices v_1, v_2, v_3 of Δ such that $v_1v_2, v_1v_3 \in \Delta_E$ and $v_2v_3 \in \Delta'_E \setminus \Delta_E$. Define $\varphi'(v_2v_3) = \varphi(v_1v_2)\varphi(v_1v_3)$ (step (7) in Figure 9). Then $\varphi'(v_2v_3)\varphi(v_1v_2)\varphi(v_1v_3) = [1]$. Every other cycle using the edge v_2v_3 has the same product value as if it would use v_1v_2 and v_1v_3 instead, hence the claim follows by definition of k-phase. For the inductive step, let $j > 1$. Again, we find vertices v_1, v_2, v_3 of Δ such that $v_1v_2, v_1v_3 \in \Delta_E$ and $v_2v_3 \in \Delta'_E \setminus \Delta_E$ and define $\varphi'(v_2v_3) = \varphi(v_1v_2)\varphi(v_1v_3)$ (step (4) – (6) in Figure 9). By the same argument as in the base case, we get that this gives us a well-defined k-phase on $(\Delta_V, \Delta_E \cup \{v_2v_3\})$. The claim follows by induction, so the base case $i = 0$ is completed.
Let now $i > 1$. Then there exist two vertices v_1 and v_2 of Δ such that they have an adjacent vertex $v_3 \in \Delta'_V \setminus \Delta_V$ and $v_1v_3, v_2, v_3 \in \Delta'_E \setminus \Delta_E$. Set $\varphi'(v_1v_3) = \varphi(v_1v_2)$ and $\varphi(v_2v_3) = [1]$ (step (2) and (3) in Figure 9). We get that $\varphi'(v_1v_3)\varphi(v_1v_2)\varphi(v_2v_3) = [1]$. Every cycle in $(\Delta_V \cup \{v_3\}, \Delta_E \cup \{v_1v_3, v_2v_3\})$ using the vertex v_3 has to use the edges v_1v_3 and v_2v_3. Hence, its product value is the same as if it would use the edge v_1v_2 instead. It follows that we got a

well-defined k-phase on $(\Delta_V \cup \{v_3\}, \Delta_E \cup \{v_1v_3, v_2v_3\})$. By induction, we are done. □

Theorem 4.46. *For any k-phase φ on C, there exist exactly $|{}^{k^*}/_{(k^*)^2}|$ enrichments at the vertices of Δ compatible with φ.*
Reciprocally, given any enrichment α at the vertices of Δ, there exists a unique k-phase φ on C such that φ is compatible with α.

Proof. Let φ be a k-phase on C. Let Δ' and φ' be as in Lemma 4.45. From Theorem 4.19 we know that there exist exactly $|{}^{k^*}/_{(k^*)^2}|$ enrichments at the vertices of Δ' compatible with φ'. As Δ is connected, we see that these are exactly the compatible enrichments for φ on Δ, too.
Reciprocally, let α be an enrichment of Δ. For v, w vertices of Δ, define $\varphi(vw) = \alpha_v \alpha_w$.
For a cycle with vertices $v_1, v_2, ..., v_n, v_1$ we get

$$
\begin{aligned}
&\varphi(v_1v_2)\cdots\varphi(v_{n-1}v_n)\varphi(v_nv_1)\\
&= (\alpha_{v_1}\alpha_{v_2})\cdots(\alpha_{v_{n-1}}\alpha_{v_n})(\alpha_{v_n}\alpha_{v_1})\\
&= (\alpha_{v_1})^2\cdots(\alpha_{v_n})^2 = [1].
\end{aligned}
$$

□

4.6.1 Equivalence of $\mathbb{R}$-phase and real phase as in [RRS22]

In [RRS22], a different formulation of real phase structure is introduced:

Definition 4.47. Let Σ be a rational polyhedral fan of pure dimension d in $\mathbb{R}^n$. A *real phase structure* $\mathcal{E}$ on Σ is a map

$$\mathcal{E}\colon \text{Facets}(\Sigma) \to \{V \subseteq \mathbb{Z}_2^d \mid V \text{an affine subspace of dimension } d\}$$

such that

(i) for every facet σ of Σ, the set $\mathcal{E}(\sigma)$ is an affine subspace of $\mathbb{Z}_2^n$ parallel to σ, in formulas, $T(\mathcal{E}(\sigma)) = T_{\mathbb{Z}_2}(\sigma)$

(ii) for every codimension one face τ of Σ with facets $\sigma_1, ..., \sigma_k$ adjacent to it, the sets $\mathcal{E}(\sigma_1), ..., \mathcal{E}(\sigma_k)$ are an even covering.

where

Definition 4.48. A collection of subsets of a set such that every element in the union is contained in an even number of the subsets is called an *even covering.*

As we have already discussed earlier, we rather interpret $\mathbb{Z}_2$ as ${}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2}$ and the affine subspaces as cosets of subgroups as defined in 4.21.
Notice that the definition of real phase given above is more general than our definition of $\mathbb{R}$-phase in the sense that the fan does not have to have codimension one.
In the paper, this definition of real phases is mainly used on fans that arise as the Bergman fan of a matroid. In particular, the fans are balanced for all weights equal to one. In this case, the $\mathbb{R}$-phase and real phase are the same in the sense of 4.32, as we will prove later.

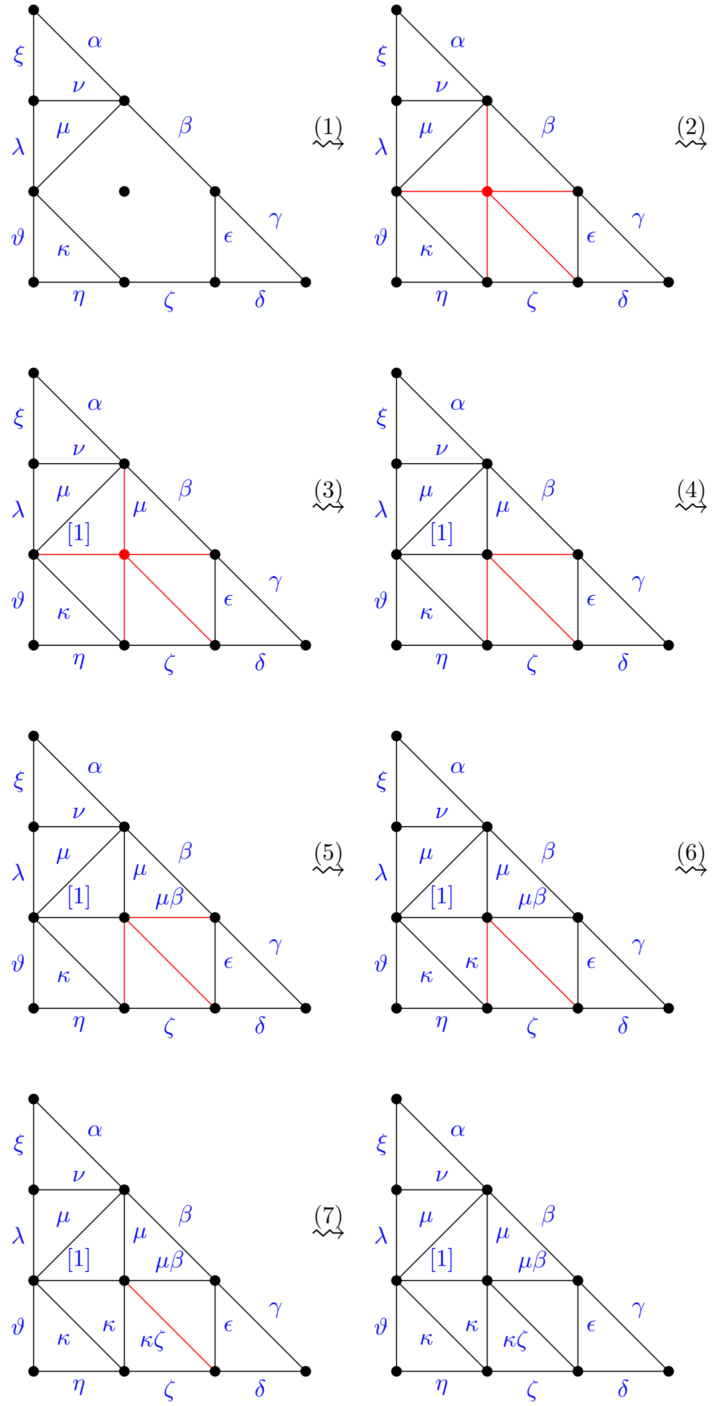


Figure 9: A short illustration of the induction in the proof of Lemma 4.45. The $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \vartheta, \kappa, \lambda, \mu, \nu$, and ξ are elements of ${}^{k^*}/_{(k^*)^2}$.

However, this definition of real phase differs from our definition of $\mathbb{R}$-phase if the fan has weights bigger than 1: As already stated earlier, the sets we associate to facets can be empty.

Example 4.49. Consider the tropical curve $V^{\text{trop}}(0 \oplus x \oplus y^2)$, see figure 10. For the $\mathbb{R}$-phase we see in the figure, the set assigned to F_1 is $\varphi_{F_1} = \{(x, y) \in {}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2} \mid [1] = x^0 y^2 = [-1]\} = \emptyset$.
We can compute $\varphi_{F_2} = \{(x, y) \in {}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2} \mid x^1 y^0 = [-1]\} = \{(-1, 1), (-1, -1)\}$ and $\varphi_{F_3} = \{(x, y) \in {}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2} \mid x^1 y^2 = x^1 y^0 = [-1]\} = \varphi_{F_2}$.

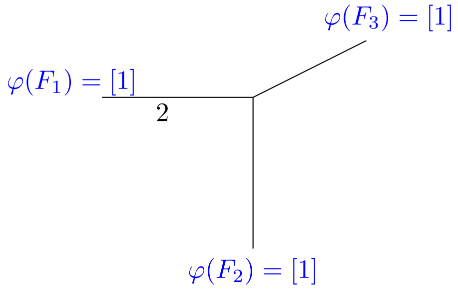


Figure 10: Example 4.49's tropical line in $\mathbb{R}^2$ with $\mathbb{R}$-phase.

Example 4.50. In the light of definition 4.47, an idea that comes up naturally is to require for a k-phase that the associated sets form a $|{}^{k^*}/_{(k^*)^2}|$-covering. However, this does not agree with our definition of k-phase: In the example given in 8 we see that $([1], [1])$ is only contained in φ_{F_3}. Moreover, we have already seen in example 4.40 that even if $|{}^{k^*}/_{(k^*)^2}| = 2$ an element can be contained in an uneven number of the sets associated to facets.

Theorem 4.51. *For a tropical fan of codimension* 1 *with all weights equal to one, the even covering property as defined in 4.47 is equivalent to the property in the definition of* $\mathbb{R}$*-phase that* $\prod_{i=1}^{k} \alpha_i = [1]$.

Proof. Let $y \in ({}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2})^n$ be a solution to the equations $x^{I_i} = -\alpha_i$ for w.l.o.g. $i = 1, ..., l$. For $i > l$ follows $x^{I_i} = \alpha_i$, as $|{}^{\mathbb{R}^*}/_{(\mathbb{R}^*)^2}| = 2$. Then

$$1 = y^0 = y^{I_1} \cdots y^{I_l} y^{I_{l+1}} \cdots y^{I_k} = [-1]^l \prod_{i=1}^{k} \alpha_i = [-1]^l,$$

which is true if, and only if, l is even. □

For fans of higher codimension however, I am not sure (yet) of how to define k-phases. One approach is to make similar constructions as in the paper [RRS22] for matroids over the (quotient) hyperfield ${}^{k^*}/_{(k^*)^2}$. This appears to work well.

5 Intersection of hypersurfaces with k-phases

After establishing k-phase for all kinds of tropical hypersurfaces, we now want to inspect what happens if we intersect tropical hypersurfaces equipped with k-phases transversely. We will then proceed with defining a k-phase structure directly on intersections.

5.1 Intersecting hypersurfaces with k-phases

Let C, C' be tropical hypersurfaces equipped with k-phases φ, φ' with transversal intersection. If we recall the intuition behind the k-phase, it is natural to define

$$\phi\colon \{F \cap F' \mid F \in \mathcal{F}(C), F' \in \mathcal{F}(C')\} \to \mathcal{P}((k^*/(k^*)^2)^n);$$
$$F \cap F' \mapsto \phi_{F \cap F'} := \varphi_F \cap \varphi'_{F'}$$

(where $\mathcal{P}((k^*/(k^*)^2)^n)$ denotes the power set of $(k^*/(k^*)^2)^n)$, as this set contains all configurations of $[x_1], ..., [x_n] \in k^*/(k^*)^2$ that allow a zero in the preimage under tropicalization. If we write $\varphi_F = \{-\alpha = x^{I-J}\}$ and $\varphi'_{F'} = \{-\alpha' = x^{I'-J'}\}$ we see that

$$\phi_{F\cap F'} = \begin{Bmatrix} -\alpha = x^{I-J} \\ -\alpha' = x^{I'-J'} \end{Bmatrix}.$$

As solutions to a system of equations in $k^*/(k^*)^2$ like in Lemma 4.25, the sets $\phi_{F\cap F'}$ are of the form yA_V for y one solution and $V = \text{span}\{I - J, I' - J'\}^\perp$.
Let $F \in \mathcal{F}(C)$ and $F \in \mathcal{F}(C')$ with corresponding vectors $I - J$, respectively $I' - J'$. Even though the intersection is transversal, we do not get any relevant information about relations between $I - J$ and $I' - J'$, as it is only relevant what they are modulo 2. An example for $I - J = I' - J' \mod 2$ is given in Example 5.2. Hence, even if C and C' are non-singular (and hence $I - J \neq 0 \mod 2$ and $I' - J' \neq 0 \mod 2$), $|\phi_{F\cap F'}|$ can be either $|k^*/(k^*)^2|^{n-2}$, $|k^*/(k^*)^2|^{n-1}$ or 0.
The next lemma lists all possible cases for $|\phi_{F\cap F'}|$ and when exactly they occur.

Lemma 5.1. *Let C, C' be two tropical hypersurfaces with k-phases φ, φ'. Let the intersection of C and C' be transversal. For F, F' facets of C, respectively C', define*

$$\phi\colon \{F \cap F' \mid F \in \mathcal{F}(C), F' \in \mathcal{F}(C')\} \to \mathcal{P}((k^*/(k^*)^2)^n);$$
$$F \cap F' \mapsto \phi_{F \cap F'} := \varphi_F \cap \varphi'_{F'}.$$

Then $\phi_{F\cap F'}$ has, for corresponding vectors $I - J$ and $I' - J'$, and $\varphi(F) = \alpha$, $\varphi'(F') = \alpha'$,

- $|k^*/(k^*)^2|^{n-2}$ *elements if, and only if,* $I - J \neq I' - J' \mod 2$ *with* $I, J \neq 0 \mod 2$.
- $|k^*/(k^*)^2|^{n-1}$ *elements if, and only if,*
  - *i)* $I - J = I' - J' \neq 0 \mod 2$ *and* $\alpha = \alpha'$ *or*
  - *ii)* $0 = I - J \neq I' - J' \mod 2$ *and* $\alpha = [1]$ *or* $0 = I' - J' \neq I - J \mod 2$ *and* $\alpha' = [1]$
- 0 *elements if, and only if,*
  - *i)* $I - J = I' - J' \neq 0 \mod 2$ *and* $\alpha \neq \alpha'$ *or*
  - *ii)* $I - J = I' - J' = 0 \mod 2$ *and* $\alpha \neq [1]$ *or* $\alpha' \neq [1]$ *or*
  - *iii)* $0 = I - J \neq I' - J' \mod 2$ *and* $\alpha \neq [1]$ *or* $0 = I' - J' \neq I - J \mod 2$ *and* $\alpha' \neq [1]$

- $|{}^{k^*}/(k^*)^2|^n$ *elements if, and only if,* $I - J = I' - J' = 0 \mod 2$ *and* $\alpha = \alpha' = [1]$.

Proof. This follows directly from Lemma 4.28. □

Example 5.2. Consider the two non-singular tropical curves C, C' with $\mathbb{R}$-phases φ, φ' in Figure 11 intersecting in the facets F and F' with corresponding vectors $(0, 1)$, respectively $(2, -1)$. We get that the blue curve has assigned the set $\{([\pm 1], [1])\}$ and the black curve $\{([\pm 1], [-1])\}$ to the intersecting facet. Hence, it is $\varphi_F \cap \varphi'_{F'} = \emptyset$.

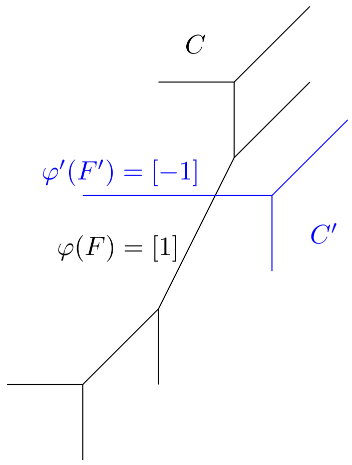


Figure 11: The intersecting curves mentioned in Example 5.2. For simplicity, the $\mathbb{R}$-phase is only noted on the relevant edges.

Example 5.3. If we consider the same curves and $\mathbb{R}$-phases as in the previous Example 5.2 and Figure 11 but change φ' to such that $\varphi'(F') = [1]$, we get the assigned sets $\varphi_F = \{([\pm 1], [-1])\} = \varphi'(F')$ and consequentially $|\varphi_F \cap \varphi'_{F'}| = 2 = 2^1 = |{}^{\mathbb{R}^*}/(\mathbb{R}^*)^2|^{2-1}$.

Example 5.4. Consider the two curves with $\mathbb{R}((t))$-phase in Figure 12. The associated set to facet F of C is $\{([\pm 1], [\pm 1]), ([\pm t], [\pm t])\}$, the one associated to the facet F' of C' is $\{(\circ, [-t])\}$. Hence, they have intersection $\{([-t], [-t])\}$ and it is $|\varphi_F \cap \varphi'_{F'}| = 1 = 4^0 = |{}^{\mathbb{R}((t))^*}/(\mathbb{R}((t))^*)^2|^{2-2}$.

As the resulting sets may have different cardinalities — even in the real and non-singular case — an approach via defining a k-phase as sets is unlikely to work. This is why we will again choose the other approach. Before going into detail there, we shortly discuss what can happen if we intersect more than just two hypersurfaces.
For $C_1, ..., C_m$ tropical hypersurfaces in $\mathbb{R}^n$ such that the intersection $D = C_1 \cap ... \cap C_m$ is transversal and k-phases $\varphi^1, ..., \varphi^m$, we again simply define

$$\phi \colon \{F_1 \cap ... \cap F_m \mid F_i \in \mathcal{F}(C_i)\} \to \mathcal{P}(({}^{k^*}/(k^*)^2)^n)$$
$$F_1 \cap ... \cap F_m \mapsto \phi_{F \cap ... \cap F_m} := \varphi^1_{F_1} \cap ... \cap \varphi^m_{F_m}.$$

With Lemma 4.25 we can say precisely how $\phi_{F \cap ... \cap F_m}$ looks like.

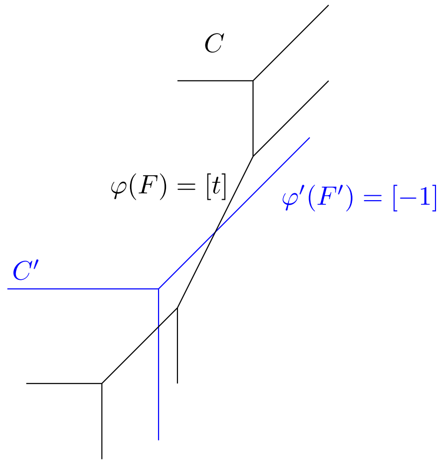


Figure 12: The intersecting curves mentioned in Example 5.4. For simplicity, the $\mathbb{R}((t))$-phase is only marked on the relevant edges.

Lemma 5.5. *For ϕ as defined above, with $I_l - J_l$ the corresponding vector to F_l for $l = 1, ..., m$, let $V = \operatorname{span}\{I_1 - J_1, ..., I_m - J_m\}$ and $p = \dim V$. Then either $|\phi_{F_1 \cap ... \cap F_m}| = 0$ or $|\phi_{F_1 \cap ... \cap F_m}| = |{}^{k^*}/{}_{(k^*)^2}|^{n-p}$.*
In the second case, $\phi_{F \cap ... \cap F_m} = yA_{V^\perp}$ holds for any $y \in \phi_{F \cap ... \cap F_m}$.
If $I_{i_1} - J_{i_1}, ..., I_{i_p} - J_{i_p}$ form a basis of V and $I_l - J_l = \lambda_{l,1}(I_{i_1} - J_{i_1}) + ... + \lambda_{l,p}(I_{i_p} - J_{i_p})$, then $\phi_{F \cap ... \cap F_m}$ is not empty, if, and only if, $\varphi^l(F_l) = (\varphi^{i_1}(F_{i_1}))^{\lambda_{l,1}} \cdots (\varphi^{i_p}(F_{i_p}))^{\lambda_{l,p}}$ holds for all $l \in \{1, ..., m\} \backslash \{i_1, ..., i_p\}$.

Proof. This follows directly from Lemma 4.25. □

5.2 k-phase for intersections

Inspired by the previous subsection and our first Definition 4.14 of a k-phase, we define

$$\phi\colon \{F \cap F' \mid F \in \mathcal{F}(C), F' \in \mathcal{F}(C')\} \to ({}^{k^*}/{}_{(k^*)^2})^2; \\ F \cap F' \mapsto \phi(F \cap F') := (\varphi(F), \varphi'(F')). \tag{7}$$

In this case, it is easier to see what happens at a codimension 1 (w.r.t. to the dimension of $C \cap C'$) face H of $C \cap C'$. As the intersection is transversal, we know that the facets adjacent to H are of the form $F_1 \cap F', ..., F_p \cap F'$ for facets $F_1, ..., F_p \in \mathcal{F}(C)$ and $F' \in \mathcal{F}(C')$ (or, of course, with the roles of C and C' exchanged). By definition of a k-phase we know that $\varphi(F_1) \cdots \varphi(F_p) = [1]$.
More generally, let $C_1, ..., C_m$ be tropical hypersurfaces in $\mathbb{R}^n$ such that their intersection $D = C_1 \cap ... \cap C_m$ is transversal. Then each codimension 1 (w.r.t. the dimension of D) face H of D is of the form $F_1 \cap ... \cap H_k \cap ... \cap F_m$ with $F_i \in \mathcal{F}(C_i)$ and H_k a codimension 1 (w.r.t. the dimension of C_k) face of C_k. In particular, if the adjacent facets of H_k are $F_k^1, ..., F_k^p \in \mathcal{F}(C_k)$, then the adjacent facets of H are $F_1 \cap ... \cap F_k^1 \cap ... \cap F_m, ..., F_1 \cap ... \cap F_k^p \cap ... \cap F_m$.
Again, we get from the definition of k-phase that $\varphi(F_k^1) \cdots \varphi(F_k^p) = [1]$. This observation inspires the next definition. For $m = 1$ it is exactly the already known definition for k-phase.

Definition 5.6. Let $C_1, ..., C_m$ be tropical hypersurfaces in $\mathbb{R}^n$ such that the intersection $D = C_1 \cap ... \cap C_m$ is transversal. A *k-phase structure* on D is a map

$$\varphi\colon \{F_1 \cap ... \cap F_m \mid F_i \in \mathcal{F}(C_i)\} \to ({}^{k^*}/{}_{(k^*)^2})^m$$
$$F_1 \cap ... \cap F_m \mapsto \varphi(F_1 \cap ... \cap F_m),$$

such that

$$\varphi(F_1 \cap ... \cap F_k^1 \cap ... \cap F_m)_k \cdots \varphi(F_1 \cap ... \cap F_k^p \cap ... \cap F_m)_k = [1]$$

holds for H a facet of D with adjacent facets $F_1 \cap ... \cap F_k^1 \cap ... \cap F_m, ..., F_1 \cap ... \cap F_k^p \cap ... \cap F_m$.

Let $C_1, ..., C_m$ be as in the definition and the intersection D be equipped with a k-phase φ. If we want to get sets of elements of $({}^{k^*}/{}_{(k^*)^2})^n$ out of this definition of k-phase, we need to solve the system of equations

$$\begin{cases} -\varphi(F_1 \cap ... \cap F_m)_1 = x^{I_1 - J_1} \\ \vdots \\ -\varphi(F_1 \cap ... \cap F_m)_m = x^{I_m - J_m} \end{cases},$$

for, as usual, $F_i \in \mathcal{F}(C_i)$ with corresponding vector $I_i - J_i$ for all $i = 1, ..., m$.

Remark 5.7. As the intersection of k-phases written as sets is just a set-theoretic intersection, the intersection is the same no matter if we intersect C_1 with C_2 or vice-versa. If we write the intersection as in (7), changing the order of intersecting only changes the order of the tuple.
For the same reason, intersecting with k-phases is associative, too.

Remark 5.8. Impractically, for stable intersections, k-phase does not work, as the following two examples show. Consider the two tropical lines from Figure 13 with stable intersection point x.
Let $\alpha = [1]$. Then we compute the set associated to x_1 to be $\{([-1], [-1])\}$ and the one to x_2 to be $\{([-1], [1])\}$. Hence, we get a different set depending on whether we shift the blue curve to the left or to the right.
For $\alpha = [-1]$ we see that the approach to only note the tuple of k-phases fails, too: For x_1 we get $([1], [1])$ but for x_2 we get $([1], [-1])$.
The third natural approach would be to just intersect all the sets associated to facets adjacent to the stable intersection point. As we have already discussed in Remark 4.39, this intersection will usually be empty. In particular, it will always be empty for non-singular hypersurfaces with a real phase.

Remark 5.9. Of course, if we intersect intersections with k-phases as defined in 5.6, we can again go through the analog process and get sets and tuples associated to facets.

6 k-phase in the context of enriched intersection theory

Starting with the enriched tropical hypersurfaces we already defined, in [PP22] is an enriched intersection multiplicity introduced. In this section, we want to

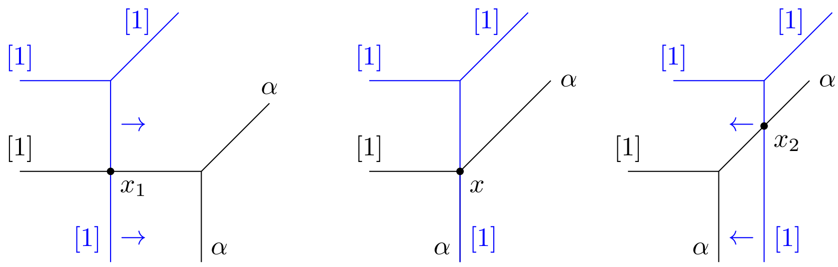


Figure 13: The intersecting lines mentioned in Remark 5.8.

study how well this concept works with k-phase. We will try to recover statements that hold for enriched hypersurfaces for our k-phase.
Before we come to the enriched intersection multiplicity, we shortly give a definition of the usual tropical intersection multiplicity.

6.1 Tropical intersection multiplicity

Let $C_1, ..., C_n$ be tropical hypersurfaces in $\mathbb{R}^n$ that intersect transversely at $p \in C_1 \cap ... \cap C_n$. Then this point corresponds to a parallelepiped P in the dual subdivision of $C_1 \cup ... \cup C_n$.
We define

$$\text{mult}_p(C_1, ..., C_n) := \text{volume of } P.$$

If the intersection point lies in $F_1 \cap ... \cap F_n$ for $F_i \in \mathcal{F}(C_i)$ and the corresponding vector of F_i is $I_i - J_i$ for all $i \in \{1, ..., n\}$, then this means

$$\text{mult}_p(C_1, ..., C_n) = |\det(I_1 - J_1, ..., I_n - J_n)|.$$

Example 6.1. Consider the intersection of two tropical lines in Figure 14. As the intersection multiplicity we compute

$$|\det \begin{pmatrix} 1-0 & 0-0 \\ 0-0 & 1-0 \end{pmatrix}| = |\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}| = 1.$$

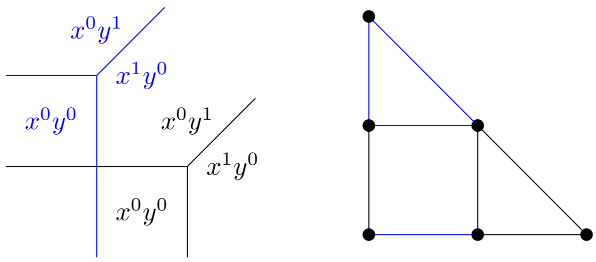


Figure 14: The intersecting lines mentioned in Example 6.1 with the dual of the union of the lines. The intersection multiplicity is 1.

Example 6.2. Consider the intersection of two tropical curves in Figure 15. We compute the intersection multiplicity to be

$$|\det \begin{pmatrix} 2-0 & 0-1 \\ 0-0 & 1-0 \end{pmatrix}| = |\det \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}| = 2.$$

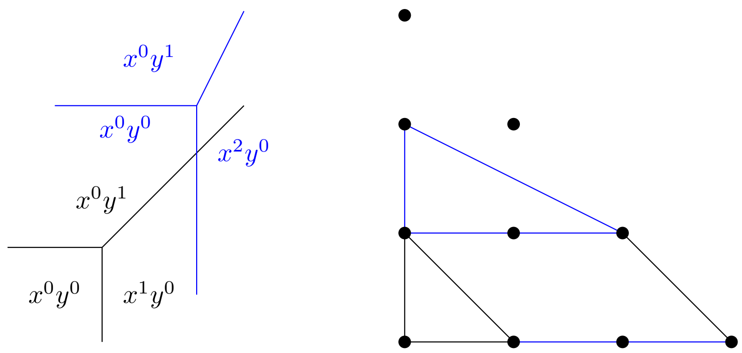


Figure 15: The intersecting curves referred to in Example 6.2 with the dual of their union. The intersection multiplicity is 2.

Example 6.3. Consider the intersection in Figure 16. The intersection multiplicity is

$$|\det \begin{pmatrix} 0-0 & 2-0 \\ 0-1 & 0-1 \end{pmatrix}| = |\det \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}| = 2.$$

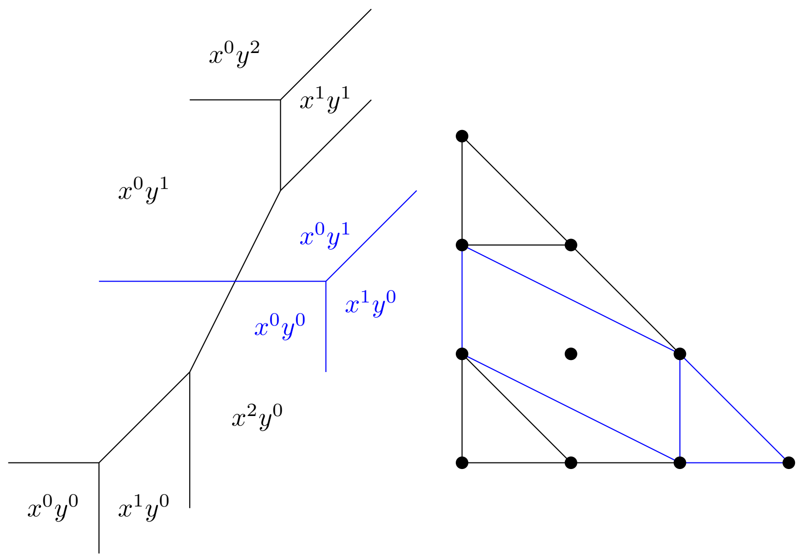


Figure 16: The intersecting curves mentioned in Example 6.3 with the dual of their union. The intersection multiplicity is 2.

6.2 The Grothendieck-Witt ring and the trace map

As the enriched intersection multiplicity takes values in the Grothendieck-Witt ring, we first need to introduce this structure.
This subsection is heavily based on [PP22]'s subsection 2.1. We start off with the important definition.

Definition 6.4 (Grothendieck-Witt ring)**.** The *Grothendieck-Witt ring* $\mathrm{GW}(R)$ of a ring R is the group completion of the semi-ring of isometry classes of non-degenerate symmetric bilinear forms over R under the direct sum $\oplus$ and tensor product $\otimes$.

For R a field with characteristic other than 2, the Grothendieck-Witt ring has a nice presentation. In this case, any form can be diagonalized, i.e. for any k-vector space V and symmetric bilinear form $\beta\colon V \times V \to k$ we can find a basis for V, such that

$$\beta((x_1, ..., x_n), (y_1, ..., y_n)) = a_1x_1y_1 + ... + a_nx_ny_n \tag{8}$$

for some $a_1, ..., a_n \in k^*$ in this basis. If we replace one of the a_i by a_ib^2 for some $b \in k^*$, the resulting form is in the same isometry class as β. Hence, the form β can be expressed as the direct sum of n symmetric bilinear forms on a one-dimensional k-vector space. Indeed, $\mathrm{GW}(k)$ is generated by the classes of bilinear forms

$$\langle a\rangle\colon k \times k \to k;\ (x, y) \mapsto axy$$

for $a \in {}^{k^*}/_{(k^*)^2}$ subject to the following relations

1. $\langle a\rangle\langle b\rangle = \langle ab\rangle$ for $a, b \in k^*$
2. $\langle a\rangle + \langle b\rangle = \langle a + b\rangle + \langle ab(a + b)\rangle$ for $a, b, a + b \in k^*$.

We define $\langle a_1, ..., a_s\rangle := \langle a_1\rangle + ... + \langle a_s\rangle$. From this follows in particular, that we can interpret ${}^{k^*}/_{(k^*)^2}$ as a subset of $\mathrm{GW}(k)$ and that the multiplication in $\mathrm{GW}(k)$ is the same as the one in ${}^{k^*}/_{(k^*)^2}$. This allows us to define multiplication of elements in ${}^{k^*}/_{(k^*)^2}$ with elements in $\mathrm{GW}(k)$ via $[a]\otimes g := \langle a\rangle g$ for $g \in \mathrm{GW}(k)$ and $[a] \in {}^{k^*}/_{(k^*)^2}$.

Definition 6.5 (Hyperbolic form)**.** We define the *hyperbolic form* h to be the form on a 2-dimensional k-vector space (or free rank 2 R-module over R when R is not a field) with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark 6.6. For any $a \in R^*$ it is $\langle a\rangle h = h$. If $R = k$ is a field of characteristic not 2, it is $h = \langle 1\rangle + \langle -1\rangle = \langle a\rangle + \langle -a\rangle$.

Definition 6.7 (Rank)**.** The *rank* of a symmetric bilinear form $\beta\colon V \times V \to R$ is defined to be the rank of the R-module V.

The rank extends to a homomorphism $\mathrm{rank}\colon \mathrm{GW}(R) \to \mathbb{Z}$.

Example 6.8. For $k = \mathbb{C}$ we have ${}^{k^*}/_{(k^*)^2} = \{[1]\}$ and hence there is only one generator $\langle 1\rangle \in \mathrm{GW}(\mathbb{C})$. It follows that $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ where the isomorphism is the rank homomorphism. In particular, results in classical enumerative geometry coincide with the counts enriched in $\mathrm{GW}(k)$ for an algebraically closed field k by taking the rank.

Example 6.9. For $k = \mathbb{R}$ we have ${}^{k^*}/_{(k^*)^2} = \{[1], [-1]\}$ and hence $\mathrm{GW}(\mathbb{R})$ has two generators $\langle 1\rangle$ and $\langle -1\rangle$. An element in $\mathrm{GW}(\mathbb{R})$ is completely determined by its rank and its signature.

Example 6.10. We have already seen that ${}^{k^*}/_{(k^*)^2} = {}^{k\{\{t\}\}^*}/_{(k\{\{t\}\}^*)^2}$ and hence $\mathrm{GW}(k) \cong \mathrm{GW}(k\{\{t\}\})$.

6.3 Enriched intersection multiplicity and k-phase

In [PP22], the authors give a definition of an enriched intersection multiplicity for n enriched tropical hypersurfaces in $\mathbb{R}^n$ which takes values in $\mathrm{GW}(k)$. To avoid even more definitions, we will define this multiplicity via a theorem from this paper instead of the original definition. However, first we still need the following definitions.

Definition 6.11 (Odd vertices)**.** We call a vertex $v = (v_1, ..., v_n) \in \mathbb{Z}^n$ in the dual subdivision *odd*, if its class equals $(1, ..., 1)$ in $\mathbb{Z}_2^n$.

Let p be an intersection point of enriched tropical hypersurfaces $C_1, ..., C_n$ and let P be the parallelepiped in the dual subdivision of $C_1 \cup ... \cup C_n$ dual to p. For $\Delta^i = I^i - J^i$ the corresponding vectors to the facets $F_i \in \mathcal{F}(C_i)$ with $p \in F_i$, this parallelepiped is given by $P = \left\{\sum_{i=1}^n K^i \mid K^i = I^i \text{ or } J^i\right\}$.
We assign a sign to each vertex of P. If $v = \sum_{i=1}^n K^i$ is a vertex of P, it can be expressed uniquely as $v = \sum_{i=1}^n I^i - \delta_{K^i,J^i}\Delta^i$, where δ_{K^i,J^i} is the Kronecker delta. The *sign* of the vertex v with respect to the parallelepiped P is defined as

$$\epsilon_P(v) := \left(\prod_{i=1}^n (-1)^{\delta_{K^i,J^i}}\right) \mathrm{sign}(\det D),$$

where D is the matrix with rows $\Delta^1, ..., \Delta^n$.

Remark 6.12. In the case $n = 2$ this means the following: If v is a corner vertex of a parallelogram P in the dual subdivision of $C_1 \cup C_2$ dual to a point $p \in C_1 \cap C_2$, it is

$$\epsilon_P(v) = \begin{cases} +1 & \text{first } C_1 \text{ then } C_2 \\ -1 & \text{first } C_2 \text{ then } C_1 \end{cases}$$

when walking around the vertex inside the parallelogram anticlockwise.

Theorem 6.13 ([PP22] Theorem 5.6.)**.** *Let p be an intersection point of enriched tropical hypersurfaces $C_1, ..., C_n$ that intersect tropically transversally at p. Let P be the parallelepiped in the dual subdivision of $C_1 \cup ... \cup C_n$ corresponding to p and let $v_1, ..., v_q$ be the odd vertices of P. If the classical intersection multiplicity $\mathrm{mult}_p(C_1, ..., C_n)$ equals m, then*

$$\widetilde{\mathrm{mult}}_p(C_1, ..., C_n) = \sum_{l=1}^q \langle \epsilon_P(v_l)\alpha_l \rangle + \frac{m-q}{2} h$$

where $\alpha_l = \Pi_{i=1}^n \alpha_{I^i}^{(l)}$ is the coefficient of the odd vertex v_l, for $l = 1, ..., q$.

Taking the rank of this enriched intersection multiplicity recovers the classical tropical intersection multiplicity.
From this theorem we get, with an easy computation, that the enriched multiplicity is defined up to a factor of $k^*/(k^*)^2$ if we fix the k-phase.

Corollary 6.14. *Let p be an intersection point of enriched tropical hypersurfaces $C_1, ..., C_n$ that intersect tropically transversally at p. Let C_1' be the enriched hypersurface we get by multiplying the enrichment α of C_1 with $\gamma \in k^*/(k^*)^2$. Then*

$$\widetilde{\mathrm{mult}}_p(C_1', ..., C_n) = \gamma\, \widetilde{\mathrm{mult}}_p(C_1, ..., C_n).$$

In particular, if we fix a k-phase on $C_1, ..., C_n$, the enriched intersection multiplicities are defined up to a factor of $k^*/(k^*)^2$.

Proof. Using the notation from Theorem 6.13 we compute

$$\begin{aligned}
&\widetilde{\text{mult}}_p(C_1', ..., C_n) \\
&= \sum_{l=1}^{q} \langle \epsilon_P(v_l) \left(\gamma \alpha_{I^1}^{(l)} \Pi_{i=2}^n \alpha_{I^i}^{(l)} \right) \rangle + \frac{m-q}{2} h \\
&= \gamma \sum_{l=1}^{q} \langle \epsilon_P(v_l) \left(\Pi_{i=1}^n \alpha_{I^i}^{(l)} \right) \rangle + \gamma \frac{m-q}{2} h \\
&= \gamma \, \widetilde{\text{mult}}_p(C_1, ..., C_n).
\end{aligned}$$

□

6.4 Enriched Bézout's theorem and k-phase

In, again, [PP22], the authors prove enriched versions of Bézout's theorem. For the first version, we get the positive answer that it holds for tropical hypersurfaces with k-phases, too.

Theorem 6.15 ([PP22] Corollary 6.5.)**.** *Let $C_1, ..., C_n$ be enriched tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $\Delta_{d_1}, ..., \Delta_{d_n}$ such that $\sum_{i=1}^n d_i \equiv n+1 \mod 2$ and assume that $C_1, ..., C_n$ intersect transversally at every intersection point. Then*

$$\sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n) = \frac{d_1 \cdots d_n}{2} h \in \text{GW}(k).$$

Corollary 6.16. *Let $C_1, ..., C_n$ be as in 6.15 and C_1' as in 6.14 the enriched hypersurface we get by multiplying the enrichment α of C_1 with $\gamma \in k^*/(k^*)^2$. Then*

$$\sum_{p \in C_1' \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) = \frac{d_1 \cdots d_n}{2} h = \sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n).$$

In particular, we get this version of Bézout's theorem for k-phases, too.

Proof. We easily compute

$$\begin{aligned}
&\sum_{p \in C_1' \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) \\
&= \sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) \\
&= \gamma \sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n) \\
&= \gamma \frac{d_1 \cdots d_n}{2} h = \frac{d_1 \cdots d_n}{2} h
\end{aligned}$$

□

In the second case however, the factor $\gamma \in {}^{k^*}/_{(k^*)^2}$ does not get "absorbed" by a hyperbolic form.

Theorem 6.17 ([PP22] Corollary 6.8.)**.** *Let $C_1, ..., C_n$ be enriched tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $\Delta_{d_1}, ..., \Delta_{d_n}$ such that $\sum_{i=1}^n d_i \not\equiv n+1 \mod 2$ and assume that $C_1, ..., C_n$ intersect transversally at every intersection point. Then*

$$\sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n) = \frac{d_1 \cdots d_n - r}{2} h + \langle a_1, ..., a_r \rangle \in \text{GW}(k)$$

where r has to be smaller or equal to the number of odd points on $\partial \Delta_{d_1 + ... + d_n}$.

Corollary 6.18. *Let $C_1, ..., C_n$ be as in 6.17 and C_1' as in 6.14 the enriched hypersurface we get by multiplying the enrichment α of C_1 with $\gamma \in {}^{k^*}/_{(k^*)^2}$. Then*

$$\sum_{p \in C_1' \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) = \frac{d_1 \cdots d_n - r}{2} h + \langle \gamma a_1, ..., \gamma a_r \rangle$$

for

$$\sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n) = \frac{d_1 \cdots d_n - r}{2} h + \langle a_1, ..., a_r \rangle \in \text{GW}(k)$$

as in Theorem 6.17.

Proof. We compute

$$\begin{aligned}
&\sum_{p \in C_1' \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) \\
&= \sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1', ..., C_n) \\
&= \gamma \sum_{p \in C_1 \cap ... \cap C_n} \widetilde{\text{mult}}_p(C_1, ..., C_n) \\
&= \gamma \left(\frac{d_1 \cdots d_n - r}{2} h + \langle a_1, ..., a_r \rangle \right) = \frac{d_1 \cdots d_n - r}{2} h + \langle \gamma a_1, ..., \gamma a_r \rangle
\end{aligned}$$

□

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